Higher-Order Unification: A structural relation between Huet's method and the one based on explicit substitutions

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Abstract. We compare two different styles of Higher-Order Unification (HOU): the classical HOU algorithm of Huet for the simply typed λ-calculus and HOU based on the $\lambda\sigma$-calculus of explicit substitutions. For doing this, first, the original Huet algorithm for the simply typed λ-calculus with names is adapted to the language of the λ-calculus in de Bruijn's notation, since this is the notation used by the $\lambda\sigma$-calculus. Afterwards, we introduce a new structural notation called *unification tree*, which eases the presentation of the subgoals generated by Huet's algorithm and its behavior. The unification tree notation will be important for the comparison between Huet's algorithm and unification in the $\lambda\sigma$-calculus whose derivations are presented into a structure called *derivation tree*. We prove that there exists an important structural correspondence between Huet's HOU and the $\lambda\sigma$-HOU method: for each (sub-)problem in the unification tree there exists a counterpart in the derivation tree. This allows us to conclude that the $\lambda\sigma$-HOU is a generalization of Huet's algorithm and that solutions computed by the latter are always computed by the former method.

Keywords: Higher-Order Unification, Calculi of Explicit Substitutions.

1 Introduction

More than thirty years ago, G. Huet [Hue75,Hue02] gave the most successful and largely used Higher-Order Unification (HOU) algorithm. HOU is undecidable [Gol81], Huet's algorithm is in fact a semi-decision procedure because it always finds the solutions to unifiable problems but may loop if the problem does not have a solution. The kernel of Huet's algorithm consists of two procedures called SIMPL and MATCH used for dealing with the so called rigid-rigid, and flexible-rigid equations, respectively, and its practical success is based on

* Corresponding author. Supported by Brazilian CAPES Foundation.

** Partially supported by Brazilian CNPq Council.

the observation that flexible-flexible equations always have solutions and consequently (for deciding whether a problem is or is not unifiable) it is not necessary to explicitly present all possible unifiers. Huet's algorithm behaves well in practice, and as a consequence, many of the modern computational systems which use HOU are based on this algorithm, such as λProlog and Isabelle/HOL. In addition, this algorithm has been extended to several higher-order equational theories [Dow01] and specialized to treat reducts of practical interest such as the case of higher-order patterns [Nip91].

The most promising alternative for treating HOU problems is based on explicit substitutions calculi and was developed over the $\lambda\sigma$-calculus almost ten years ago [DHK00]. This alternative method has been shown to be of general applicability for other calculi of explicit substitutions such as λs_e-calculus [ARK01]. Calculi of explicit substitutions are essentially formal mechanisms attempting to solve an important drawback of the λ-calculus: the implicitness of substitution, that is the basic operation on which the computational functionality of the λ-calculus is founded. The formal basis of some programming languages is founded on explicit substitutions; for instance, λProlog is founded on the suspension calculus of explicit substitutions. As a matter of fact, real programming languages are based on some *ad-hoc* (and mostly obscure) explicit implementation of the substitution operation. When substitution is made explicit, it allows one to include HOU mechanisms at a lower level; that is, directly over the associated language of explicit substitutions instead of implementing HOU mechanisms, as usual, as strategies in a higher level of the implementation based on Huet's algorithm. The importance of a precise knowledge of the style of explicit substitutions used in the implementation of programming languages has been made evident recently in [LNQ04]. In that work, the efficiency of different implementations of λProlog over the system Teyjus was tested for several programs. Simple changes in the way explicit substitutions are treated over the suspension calculus were shown to imply great changes in the performance of the language.

Essentially, HOU via calculi of explicit substitutions consists of, firstly, translating HOU problems presented in the language of the simply typed λ-calculus (in de Bruijn's notation) to the language of the explicit substitutions calculus; this process is known as a *precooking translation*. Afterwards, precooked problems are resolved as first-order unification problems modulo the equational theory which defines the calculus of explicit substitutions and, finally, the solutions are translated back to the language of the original problem (see the Fig. 1, that has been taken from [ARK03]). Therefore, the main advantage of the use of explicit substitutions to solve HOU problems is that the substitution operation becomes a first order substitution (called *grafting*) and the higher-order substitutions which are solutions of the original problem can be obtained by applying the inverse of the precooking translation to the generated graftings, i.e., to the solutions of the precooked version of the original problem.

In [DHK00], it has been noted that the $\lambda\sigma$-HOU algorithm is a generalization of Huet's method. In this paper, we refine this result by establishing a structural relation between sub-problems in the λ-calculus and in the $\lambda\sigma$-calculus in the

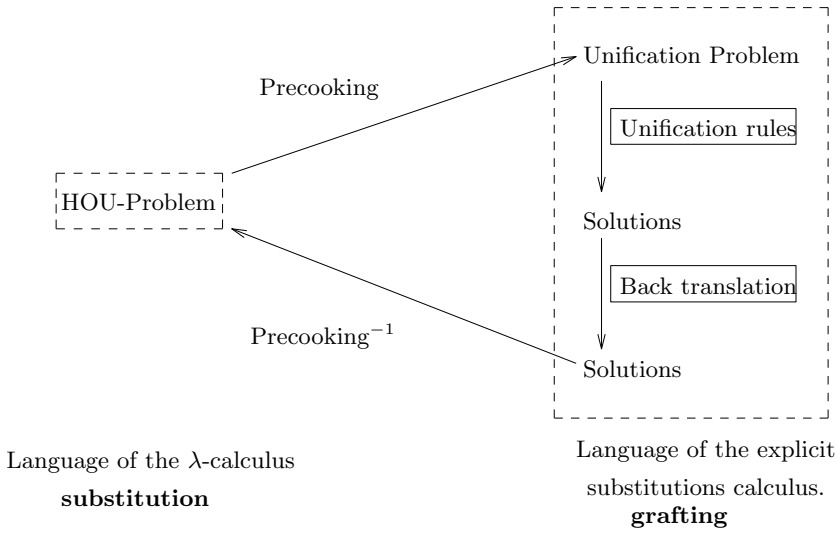


Fig. 1. HOU via calculi of Explicit substitutions.

following way: we introduce a new notation called *unification tree* which clarifies the presentation of Huet's algorithm and eases the presentation of subgoals generated by Huet's algorithm because each step of the algorithm is represented by an arc in the tree. This notation is independent of the grammar used and can be used for both λ-calculi with names or in de Bruijn's notation. In a similar way, applications of the unification procedure in the simply typed $\lambda\sigma$-calculus are represented as trees, called *derivation trees*.

We prove that for a given unification problem P in the simply typed λ-calculus in de Bruijn's notation, each subgoal (or subproblem) generated in the unification tree of P has a counterpart in the derivation tree of its precooking translation P_F in the $\lambda\sigma$-calculus, i.e., there exists a structural relation between the unification tree of P and the derivation tree of P_F. From this, we establish a relation between the solutions of subproblems of P and subproblems of P_F.

In section 2 we present the simply typed λ-calculus with names, Huet's algorithm and the unification tree notation. In section 3 we define the simply typed λ-calculus and we introduce Huet's algorithm in de Bruijn's notation. A detailed description of Huet's algorithm is given and the relevant aspects that differ from the presentation with names are emphasized with examples. In section 4 we briefly present the $\lambda\sigma$-HOU method and we formalize the relation between unification in the simply typed λ-calculus and in the simply typed $\lambda\sigma$-calculus by relating unification trees and derivation trees. Finally, in the last section, we conclude and give directions for future work.

2 Background

Since the λ-calculus with names is much better for human reading than the λ-calculus in de Bruijn's notation, we start the next subsection with a general presentation of the simply typed λ-calculus with names and of Huet's algorithm. For this presentation we use standard notations and suppose familiarity with basic notions on rewriting theory [BN98], type theory [Hin97] and λ-calculus [Bar84].

2.1 Simply typed λ-calculus with Names

The λ-terms (firstly without types) are built over constants and bound variables represented by $x, y, z, \ldots$ which range over the set $\mathcal{V}$, meta-variables (free variables) $X, Y, Z, \ldots$ which range over the set $\mathcal{X}$, applications and abstractions that are inductively defined by:

$$a \ ::= \ x \mid X \mid a\ a \mid \lambda_x.a, \text{ where } x \in \mathcal{V} \text{ and } X \in \mathcal{X}.$$

Remark 1. As usual, parenthesis are used to avoid ambiguities and we assume that applications are left associative, i.e., $(a_1\ a_2 \ldots a_n)$ means $((\ldots(a_1\ a_2)\ldots)\ a_n)$ and, abstractions are right associative, i.e., $\lambda_{x_1}\lambda_{x_2}\ldots\lambda_{x_n}.u$ is interpreted as $\lambda_{x_1}.(\lambda_{x_2}.(\ldots(\lambda_{x_n}.u)\ldots))$. Moreover, an application has higher priority then an abstraction. In this way, $\lambda_x.a\ b$ means $\lambda_x.(a\ b)$.

Remark 2. The separation of constants and bound variables on one side and, meta-variables (also known as unification variables) on the other side is important to distinguish between the substitutions generated by β-reductions and those generated by the unification procedure. In fact, bound variables and constants are not concerned with the unification process and the meta-variables will play the role of the unification variables.

In the λ-calculus with names, terms are interpreted modulo α-conversion, which means that bound variable names used in abstractions are irrelevant. For example, $\lambda_x.x\ z$ and $\lambda_y.y\ z$ represent the same λ-term. Free and bound occurrences are defined as usual; for instance, in the term $(\lambda_y.y\ z)\ y$, z and the second occurrence of y are free while the first occurrence of y is bound.

The basic operations of the λ-calculus are the β-reduction and η-reduction[4]. The former, implements the applications of functional terms over arguments and the latter represents the functional equivalence. They are "implicitly" defined as:

$(\lambda_x.a)\ b \to a\{x/b\}$ (β)

$\lambda_x.a\ x \to a$, if x does not occur free in a. (η)

In (β), "$a\{x/b\}$" represents the term obtained from a by substituting all its free occurrences of x for b. Implicitness of the definition of β-reduction is a consequence of this pseudo-definition of substitution. And this is the main

[4] We will use the word "reduction" for both the β and η rewriting rules (usually called β- and η-contraction) and the rewriting relation generated from these rules.

theoretical drawback of the λ-calculus, when it has to be used for concrete implementations. In fact, for implementing the λ-calculus one has to decide how to implement substitutions and this is done usually by *ad-hoc* mechanisms, which are adjusted during the implementation process. Calculi of explicit substitutions attack this problem attempting to formalize, in different styles, the notion of substitution, which make these formalism close to concrete implementations.

The η-reduction stands for the functional equivalence and this can be understood by noticing that, whenever it applies, for any term b, it holds that $a\ b$ and $(\lambda_x.a\ x)\ b$ coincide since $(\lambda_x.a\ x)\ b \to_\beta (a\ x)\{x/b\} = a\ b$.

Notations used for rewriting concepts of the λ-calculus with names as well as for any other rewriting system in this work are the standard ones from rewriting theory (see [BN98,Hin97]). Let a be a λ-term, a β-*redex* in a is a sub-term of a which is an instance of the left hand side of the β-reduction rule. The right hand side of an instance of the β-reduction rule is called a β-*contractum*. Supposing the term obtained by replacing in a the β-redex by its contractum is the term b, we write $a \to_\beta b$. A term without β-redexes is said to be in β-*normal form* or β-nf for short. The inverse of the binary relation $\to_\beta$ is denoted by ${}_\beta\!\leftarrow$ and its reflexive transitive closure by $\to_\beta^*$. The symmetric closure of $\to_\beta$ which is the relation $\to_\beta \cup_\beta \leftarrow$ is denoted by $\leftrightarrow_\beta$ and its reflexive transitive closure, called β-*conversion*, by $=_\beta$. A β-nf of a term a is a term b such that b is a β-nf and $a \to_\beta^* b$. Similarly, we define η-redexes, η-contractum, the notations $\to_\eta$, η-conversion, η-nf, etc. Also for the relation $\to_\beta \cup \to_\eta$, denoted as $\to_{\beta\eta}$, the same notations are used.

It is well known that the adequate environment for higher-order unification is the simply typed λ-calculus. In the following we present the simply typed version of the λ-calculus with names. We start assuming that there exists an infinite set $\mathbb{T}$ of type variables also known as *atomic types*. Types are inductively defined by:

$$A ::= K \mid A \to A$$

where K ranges over the set $\mathbb{T}$. We say that A is the *target type* of the type $A_1 \to \ldots \to A_n \to A$, where $n \geq 0$. We follow the Church approach for typing terms. In this approach, differently to Curry one (also known as type assignment theory), typed λ-terms are inductively defined by:

$$a ::= x \mid X \mid a\ a \mid \lambda_{x:A}.a, \text{ where } x \in \mathcal{V} \text{ and } X \in \mathcal{X}.$$

A type assignment is an expression of the form $a : A$, where a is a λ-term and A is a type. Type contexts, or just contexts, are used to store the type information of the constants occurring in a term and are defined as finite sets of type assignments. We use $\Gamma, \Delta, \ldots$ to denote contexts. A context Γ is said to be *consistent* if each variable in Γ has no more than one assignment. We assume contexts to be consistent and use the following typing rules:

(var) $\dfrac{}{x : A \vdash x : A}$

(meta) $\dfrac{}{\Gamma \vdash X : A}$, where Γ is any context.

(app) $\dfrac{\Gamma_1 \vdash a : A \to B \quad \Gamma_2 \vdash b : A}{\Gamma_1 \cup \Gamma_2 \vdash (a\ b) : B}$, if $\Gamma_1 \cup \Gamma_2$ is consistent.

(lambda) $\dfrac{\Gamma \vdash a : B}{\Gamma - x \vdash \lambda_{x:A}.a : A \to B}$, if Γ is consistent with $x : A$

The *type judgment* $\Gamma \vdash a : A$ is said to be *derivable* if it can be deduced from the above typing rules. In the rule (lambda), the notation $\Gamma - x$ means that the assignment to x in Γ (if it exists) is removed and, the condition "Γ is consistent with $x : A$" means that either Γ contains $x : A$ or Γ contains no assignment to x at all. In the former case, we say that x is discharged from Γ and in the latter case that x is discharged vacuously from Γ.

The rule (meta) implies that the type of meta-variables are independent from the context, which is necessary for placing repetitions of meta-variables at different levels of abstraction in λ-terms. For instance, consider the type judgment

$$\vdash \lambda_{z:A\to(A\to A)\to A}.z\ Y\ \lambda_{x:A}.Y : (A \to (A \to A) \to A) \to A$$

This judgment is derivable from the above typing rules after applying the (meta) rule twice for obtaining the judgments

$$\vdash Y : A \text{ and } x : A \vdash Y : A$$

A λ-term a is called *well typed* if and only if there exists a context Γ and a type A, such that $\Gamma \vdash a : A$ is derivable. It is well known that the λ-calculus restricted to well typed terms is closed under sub-terms and $\beta\eta$-reduction. Moreover, it is strongly terminating, which means that every $\beta\eta$-reduction starting from a well typed λ-term is finite.

2.2 Huet's Algorithm

In the next paragraphs we give a general overview of Huet's algorithm [Hue75]. Roughly speaking, Huet's algorithm is a semi-decision procedure for unification in the simply typed λ-calculus. It is semi-decision because it always finds solutions to unifiable unification problems but may loop if the unification problem has no solution. We start the presentation with some relevant definitions.

Definition 3 (Structure of nfs). *If a is well typed and in β-nf, then it has the form*

$$\lambda_{x_1:A_1} \dots \lambda_{x_n:A_n}.h\ e_1 \dots e_p$$

where $n, p \geq 0$, h is a constant, a bound variable or a meta-variable, called the head *of a, and $e_1, \dots, e_p$ are λ-terms in β-nf, called the* arguments *of h. The $\lambda_{x_1:A_1}, \dots, \lambda_{x_n:A_n}$ are the* external abstractors *of a and $\lambda_{x_1:A_1} \dots \lambda_{x_n:A_n}.h$ is called its* heading.

Definition 4. *A λ-term in β-nf is* rigid *if its head is a constant or a bound variable. Otherwise, the term is* flexible*, i.e., if its head is a meta-variable.*

Definition 5 (η-long nf ([Hin97])). *A well typed λ-term a in β-nf is in η-long normal form, written η-lnf for short, if every variable occurrence in a is followed by the longest sequence of arguments allowed by its type; i.e., if each component of the form $(u\ e_1 \dots e_p)$ with $p \geq 0$ that is not in function position has an atomic type.*

From now on, "λ-terms" is a short hand for "well typed λ-terms" and, we assume that terms are always in η-lnf.

Definition 6 (Unification Problem). *A* unification problem *P in the simply typed λ-calculus is a conjunction of equations of the form $a =^? b$, where a and b are two λ-terms in η-lnf of the same type, all terms of the problem in the same context, say Γ. In this case, we say that P is well typed in context Γ. The equation $a =^? b$ is called* rigid-rigid *(resp.* flexible-flexible*) if both a and b are rigid (resp. flexible) terms, and* flexible-rigid *if a is flexible and b is rigid or vice-versa. An equation of the form $a =^? a$ is called* trivial*.*

The requirement for a general unique context in unification problems arises from the necessity to give the same assignments for constant names occurring in different equations of the unification problem, as illustrated by the following example.

Example 7. Let $\Gamma = \{x : A, f : A \to A\}$. Consider the unification problem

$$X(f\ x) =^? f\ x \wedge X(f\ x) =^? f(X\ x)$$

Notice that the type judgments $\Gamma \vdash x : A$, $\Gamma \vdash f : A \to A$ and $\Gamma \vdash X : A \to A$ are derivable. Consequently, all terms involved in the equations of this problem have type A in context Γ.

HOU is undecidable[Gol81], nevertheless Huet[Hue75] developed a semi-decision algorithm that finds a solution if it exists and may loop if it does not. This semi-decision algorithm, known as Huet's algorithm, is based on two procedures called SIMPL and MATCH. The procedure SIMPL is used for simplifying rigid-rigid equations while the procedure MATCH incrementally generates substitutions for flexible-rigid equations that will compose the solutions of the original problem. Flexible-flexible equations always have solutions and, by this reason Huet's algorithm does not need to deal with them. This is why Huet's algorithm is also known as a pre-unification algorithm. In the following we give some intuition on how it works.

Let Γ be a context and P a unification problem well typed in context Γ. The first step of Huet's algorithm is a simplification step, i.e., an application of SIMPL that consists in "breaking" rigid-rigid equations (that have the same heads) into "smaller equations" that need to be solved. For instance, suppose

that P is a unification problem containing the following rigid-rigid equation well typed in context Γ:

$$\lambda_{x_1:A_1} \ldots \lambda_{x_n:A_n}.h\ e_1^1 \ldots e_p^1 =^? \lambda_{y_1:A_1} \ldots \lambda_{y_n:A_n}.h\ e_1^2 \ldots e_p^2 \tag{1}$$

where $n, p \geq 0$, h is either a bound variable or a constant and $e_1^1 \ldots e_p^1, e_1^2 \ldots e_p^2$ are terms in β-nf. Notice that the number of external abstractors must be the same because, by definition, the terms in the left and right hand side of the equation have the same type and are in η-lnf. An application of SIMPL to P will replace the equation (1) by the following conjunction of equations:

$$\begin{gathered} \lambda_{x_1:A_1} \ldots \lambda_{x_n:A_n}.e_1^1 =^? \lambda_{y_1:A_1} \ldots \lambda_{y_n:A_n}.e_1^2 \\ \wedge \ldots \wedge \\ \lambda_{x_1:A_1} \ldots \lambda_{x_n:A_n}.e_p^1 =^? \lambda_{y_1:A_1} \ldots \lambda_{y_n:A_n}.e_p^2 \end{gathered} \tag{2}$$

which are well typed in context Γ.

The application of SIMPL to rigid-rigid equations with different heads returns a failure status because the current problem is not unifiable. This simplification step is repeated for all rigid-rigid equations of the current unification problem and, as a consequence, a simplified problem contains only flexible-rigid and/or flexible-flexible equations. Trivial equations are automatically eliminated during the whole process.

Example 8. Let $\Gamma = \{w : A, u : A \to B, v : A \to A\}$ be a context, X a meta-variable of type $A \to B$ and consider the unification problem composed by the sole rigid-rigid equation:

$$\lambda_{y:B\to B}.y\ (X\ w) =^? \lambda_{x:B\to B}.x\ (u\ (v\ w))$$

which is well typed in context Γ. An application of SIMPL to this problem generates the following simplified unification problem:

$$\lambda_{y:B\to B}.X\ w =^? \lambda_{x:B\to B}.u\ (v\ w)$$

which is well typed in context Γ.

For each simplified unification problem containing at least one flexible-rigid equation, Huet's algorithm calls the procedure MATCH. The procedure MATCH receives a flexible-rigid equation as argument and returns a finite set Σ of substitutions for the head of the flexible term of the given equation. The substitutions generated by MATCH are based on two rules called *imitation* and *projection*. To explain how these rules work, let Γ be a context, and consider the following flexible-rigid equation:

$$\lambda_{x_1:A_1} \ldots \lambda_{x_n:A_n}.X\ e_1^1 \ldots e_{p_1}^1 =^? \lambda_{y_1:A_1} \ldots \lambda_{y_n:A_n}.h\ e_1^2 \ldots e_{p_2}^2 \tag{3}$$

well typed in context Γ, where

- $n, p_1, p_2 \geq 0$;
- X is a meta-variable of type $B_1 \to \ldots \to B_{p_1} \to A$ (A atomic);
- h is either a bound variable or a constant of type $C_1 \to \ldots \to C_{p_2} \to A$ (A atomic);
- if $p_1 \neq 0$ then e_i^1 is a λ-term in η-lnf of type B_i for all $1 \leq i \leq p_1$;
- if $p_2 \neq 0$ then e_j^2 is a λ-term in η-lnf of type C_j for all $1 \leq j \leq p_2$.

Imitation Rule. The imitation rule generates a substitution that replaces X, the head of the flexible term, by another term whose head corresponds to the head of the rigid term of the current equation, i.e., by a term with head h (consider the equation (3)). In this sense it tries to imitate the term on the right hand side of the equation. Imitation is possible only if the head of the rigid term of the considered equation is a constant due to the fact that variable capture is forbidden in the λ-calculus. Then, if h is a constant, the imitation substitution generated is given by:

$$X/\lambda_{z_1:B_1} \ldots \lambda_{z_{p_1}:B_{p_1}}.h\ (H_1\ z_1 \ldots z_{p_1}) \ldots (H_{p_2}\ z_1 \ldots z_{p_1}) \tag{4}$$

where, if $p_2 > 0$ then H_i is a fresh meta-variable of type $B_1 \to \ldots \to B_{p_1} \to C_i$ for each $1 \leq i \leq p_2$. Of course, if h has atomic type, i.e., if $p_2 = 0$ then no meta-variable is introduced by the previous substitution and, the imitation substitution is given by $X/\lambda_{z_1:B_1} \ldots \lambda_{z_{p_1}:B_{p_1}}.h$.

Example 9. Consider the flexible-rigid equation generated in Example 8. An imitation substitution is possible because the head u of the rigid term is a constant. This imitation substitution is given by:

$$X/\lambda_{z:A}.u\ (H_1\ z)$$

where H_1 is a fresh meta-variable of type $A \to A$. Note that the above substitution is not a solution of the original problem but, it is part of a possible solution. In fact, substitutions generated by Huet's algorithm are incrementally generated in the sense that each application of MATCH determines only part of the solution. At the end of the unification process, the composition of all the substitutions along a success branch comprises a solution of the original problem (see Fig. 2).

Projection Rule. A projection is a substitution generated when the head h of the rigid term is either a bound variable or a constant. A projection means that the head X of the flexible term is "projected" over its arguments. Considering equation (3), X can be projected over the arguments that have the same target type as X. Since X has of type $B_1 \to \ldots \to B_{p_1} \to A$, suppose that e_i^1 has type $B_i = D_1 \to \ldots \to D_q \to A$ for some $i = 1, \ldots, p_1$ and where $q \geq 0$. In this case, the projection substitution is given by:

$$X/\lambda_{z_1:B_1} \ldots \lambda_{z_{p_1}:B_{p_1}}.z_i\ (H_1\ z_1 \ldots z_{p_1}) \ldots (H_q\ z_1 \ldots z_{p_1})$$

where, if $q > 0$ then H_j is a fresh meta-variables of type $B_1 \to \ldots \to B_{p_1} \to C_j$ for all $1 \leq j \leq q$.

As a last remark, notice that there exists at most one possible imitation and p_1 possible projections for a given flexible-rigid equation. In case no substitution is generated, i.e., if Σ is the empty set then Huet's algorithm stops reporting a failure status because the current unification problem (and therefore the original unification problem) is not unifiable.

Example 10. Consider again the flexible-rigid equation generated in Example 8. In this case, no projection is possible because the target type of X is B and the target type of its sole argument w is A.

Calls of SIMPL and MATCH are synchronized by the main procedure of Huet's algorithm. The main procedure receives a unification problem and, if it contains a rigid-rigid equation, it calls SIMPL. In case the original problem does not contain a rigid-rigid equation or after a possible application of SIMPL to it, the main procedure will look for a flexible-rigid equation in the current problem. If such equation exists, the procedure MATCH is applied to this equation. Otherwise, it is a conjunction of flexible-flexible equations and, in this case, the algorithm stops and reports a success status. After an application of MATCH, either terminals or new unification problems are generated and in the latter case, this process is repeated for each of the new generated unification problems.

Since HOU is undecidable, there exist unification problems for which Huet's algorithm does not terminate (cf. [Hue75]). The application of Huet's algorithm can be seen into a tree structure, called *matching tree*, presented in [Hue75]. The matching tree is a tree whose nodes are labeled with simplified unification problems or terminals (Success or Fail) and linked to a finite number of successors by arcs labeled with substitutions. The next example shows the matching tree for the problem presented in Example 8.

Example 11. The matching tree for the problem presented in Example 8 is given in Fig. 2. The root of the tree contains the simplified version of the original problem and the arc starting in it corresponds to an imitation substitution generated after an application of MATCH. The following node contains the simplified problem obtained after the application of this substitution. A new application of MATCH to this new unification problem generates two substitutions: an imitation that leads to two success nodes and, a projection that leads to a fail node. The solutions of the original problem are obtained by composing the substitutions generated along a success node. In this case, the solutions are given by $X/\lambda_{z:A}.u\ (v\ w)$ and $X/\lambda_{z:A}.u\ (v\ z)$.

Example 12. (Continuing example 7) Notice that the sole solution of the unification problem $X(f\ x) =^? f\ x \wedge X(f\ x) =^? f(X\ x)$ is the identity function: $X/\lambda_{z:A}.z$. The solutions for the second equation include the identity function and all compositions of f: $X/\lambda_{z:A}.f\ z, X/\lambda_{z:A}.f(f\ z), X/\lambda_{z:A}.f(f(f\ z)), \ldots$

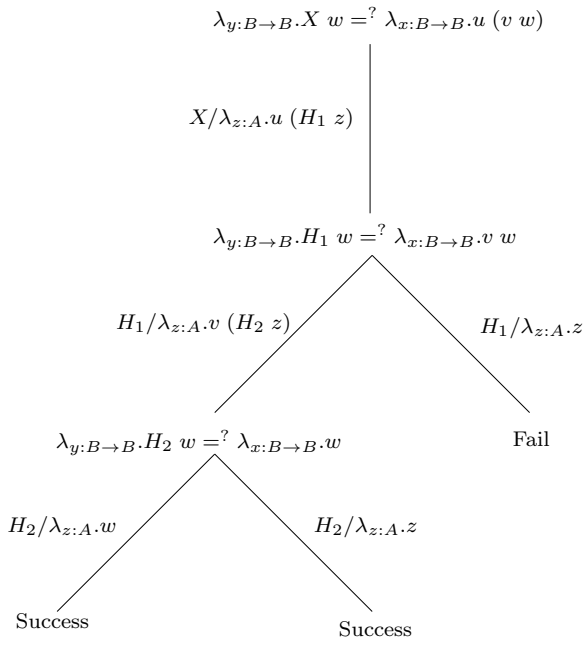


Fig. 2. The Matching Tree

2.3 Unification Tree Notation

In this subsection, we introduce the *unification tree* notation for giving a systematic presentation of Huet's algorithm over the simply typed λ-calculus. Using this structure we can exhibit the connection between the two main procedures of Huet's algorithm naturally. This clarifies the description and simplifies the comparison between explicit substitutions based HOU procedures and Huet's method.

The unification tree notation derives from Huet's matching tree [Hue75] by adding new arcs for applications of SIMPL and labels for the unification problems and substitutions. These labels provide information about the position of the unification problems and of the substitutions in the unification tree (see Fig. 3).

A unification tree $\mathcal{A}(P)$ for a given unification problem P is built as follows:

1. Label P with the subscript ϵ (the empty position), i.e., P_ϵ. This subscript means that this problem is in the root of the unification tree.
2. For a node labeled with P_α, its child node is written $\overline{P_\alpha}$ whenever it is obtained by an application of SIMPL. This step is represented by a curly line in the unification tree since the subscript remains the same after a simplification step.
3. For a node labeled with P_α containing the flexible-rigid equation eq, call $\sigma_{\alpha 1}$, $\sigma_{\alpha 2}$, ..., $\sigma_{\alpha k}$ $(k > 0)$ the substitutions generated by an application of

MATCH to *eq*. The children nodes of P_α, written $P_{\alpha 1}, \ldots, P_{\alpha k}$ are defined by $P_{\alpha i} := \overline{P_\alpha}\sigma_{\alpha i}$, for all $1 \leq i \leq k$.

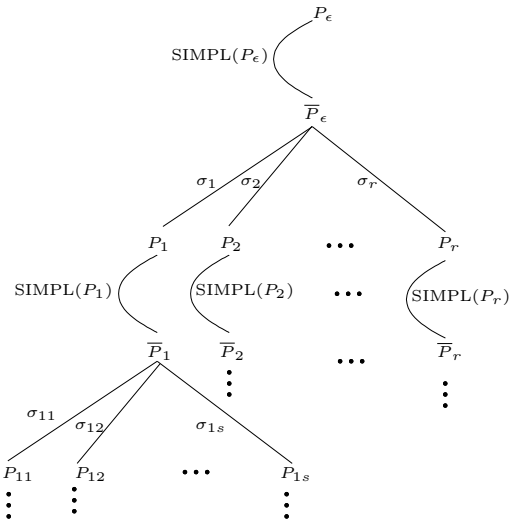


Fig. 3. The Unification Tree

Using this notation, it is straightforward to see for instance that, for a given higher-order unification problem P, a substitution with label σ_{12315} is generated (by an application of MATCH) from a unification problem with label P_{1231}. The solutions of unification problems can be easily computed by composing the generated substitutions from the root of the unification tree to a success leaf. For instance, if P_{1223} is a success node but P_{122} is not, then the substitution solution corresponding to this success path is given by the composition $\sigma_1\sigma_{12}\sigma_{122}\sigma_{1223}$. In subsection 3.2, we describe Huet's algorithm in de Bruijn's notation using the unification the tree notation.

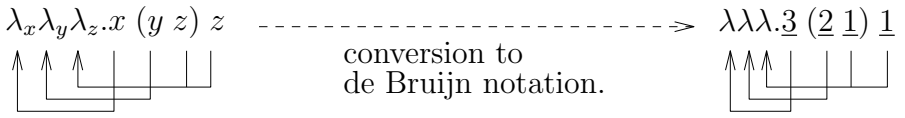
3 The Simply Typed λ-calculus and Huet's Algorithm in de Bruijn's Notation

3.1 Simply Typed λ-calculus in de Bruijn's Notation

In this subsection, we present the simply typed λ-calculus in de Bruijn's notation [dB72]. The philosophy of de Bruijn's notation is based on the fact that the link between a bound variable and the corresponding λ in a term, which binds this x (we also say that x is bound by λ_x), could also be indicated by the binding height of an occurrence. To do so, bound variables and constants are

represented by positive integers called *de Bruijn indexes*, which range over $\mathbb{N} = \{1, 2, \ldots\}$ and free variables (or meta-variables) are represented by capital letters $X, Y, Z, \ldots$, which range over the set $\mathcal{X}$. Meta-variables were not used in the original presentation of de Bruijn, but this separation of variables in two different classes is important for two reasons. First, for a better understanding of the unification methods presented here because we keep a clear distinction between the substitutions generated by the unification procedure and the ones generated by β-reductions, and second because we continue with a grammar for terms that is similar to the one used in the presentation of the λ-calculus with names (see section 2.2) which permits a better comprehension of the similarities and differences between the two approaches.

Rewriting λ-terms with names to de Bruijn's notation is an easy task. Consider, for instance, the closed term $\lambda_x\lambda_y\lambda_z.x\ (y\ z)\ z$. Translating this term to de Bruijn's notation consists in replacing each variable by the number that corresponds to the height of the abstractor that binds it:



Contexts for the λ-calculus in de Bruijn's notation are represented by a list of types. In the presentation with names, contexts were just finite sets of assignments. They now need to be ordered because constants, that we also call *free de Bruijn indexes*, refer to a specific position in the context.

Well typed λ-terms in the λ-calculus with names can be translated to de Bruijn's notation by fixing a referential containing its constants. So, suppose we want to write $a = \lambda_{xyz}.y\ (X\ u\ z)\ v$ in the referential u, v. To do so, we consider the term a in the scope of the abstractors $\lambda v\lambda u$, i.e., $\lambda_{vuxyz}.y\ (X\ u\ z)\ v$ which gives $\lambda\lambda\lambda.\underline{2}\ (X\ \underline{4}\ \underline{1})\ \underline{5}$. Note that, using the referential v, u we get $\lambda\lambda\lambda.\underline{2}\ (X\ \underline{5}\ \underline{1})\ \underline{4}$. In general, to convert a λ-term with names to its counterpart in de Bruijn notation we need to create a referential containing all its free variables and, as shown in the above example, different referentials lead to different de Bruijn λ-terms. Notice that meta-variables remain unchanged during this translation. For typed terms, such referential corresponds to a context.

Definition 13. *The set of untyped λ-terms in de Bruijn's notation is defined inductively by:*

$$a ::= \underline{n} \mid X \mid a\ a \mid \lambda.a \qquad \textit{where } n \in \mathbb{N} \textit{ and } X \in \mathcal{X}.$$

We define the syntax of simply typed λ-calculus in de Bruijn's notation by:

Types	$A ::= K \mid A \to A$	*where* $K \in \mathbb{T}$.
Contexts	$\Gamma ::= nil \mid A \cdot \Gamma$	
Terms	$a ::= \underline{n} \mid X \mid a\ a \mid \lambda_A.a$	*where* $n \in \mathbb{N}$ *and* $X \in \mathcal{X}$.

We write $\Lambda_{dB}(\mathcal{X})$ for the set of simply typed λ-terms in de Bruijn's notation. The typing rules are as follows:

$$\textit{(var)}\quad \frac{}{A\cdot\Gamma\vdash \underline{1}:A} \qquad \textit{(var+)}\quad \frac{\Gamma\vdash \underline{n}:B}{A\cdot\Gamma\vdash \underline{n+1}:B}$$

$$\textit{(lambda)}\ \frac{A\cdot\Gamma\vdash a:B}{\Gamma\vdash \lambda_A.a:A\to B} \qquad \textit{(app)}\quad \frac{\Gamma\vdash a:A\to B\ \ \Gamma\vdash b:A}{\Gamma\vdash\ (a\ b):B}$$

In addition, to each meta-variable X we associate a unique type A and, we assume that for each type there exists an infinite number of meta-variables with that type. We add the following type rule for meta-variables:

$$\textit{(meta)}\ \frac{}{\Gamma\vdash X:A}, \qquad \textit{where } \Gamma \textit{ is any context.}$$

As in the λ-calculus with names, the type of meta-variables is independent from its context. Nevertheless, the type of λ-terms (that contain constants) depends on the context. In addition, if a is a λ-term, we write a^{Γ}_{A} as a short hand for the type judgment $\Gamma\vdash a:A$.

Definition 14 ([DHK00]). *Let $n\geq 0$, $A_1,\ldots,A_n$ be types and Γ and Δ be two contexts. We say that Γ is an* extension *of Δ if it has the form $\Gamma = A_1\cdot\ldots\cdot A_n\cdot\Delta$. It is a strict extension if $n\neq 0$.*

The β-reduction for λ-terms in de Bruijn's notation is given by:

$$(\lambda_A.a)\ b\to a\{\underline{1}/b\} \qquad (\beta)$$

We say that a λ-term a, in de Bruijn's notation, is in β-normal form (β-nf for short) if a does not have a sub-term of the form $(\lambda_A.b)\ c$. This definition of β-reduction requires specific rules for propagating the substitution $\{\underline{1}/b\}$ over the term a. This is done by the following definition:

Definition 15. *Let $\underline{n}, a, b$ be well typed λ-terms in de Bruijn's notation such that $\underline{n}$ and a are two λ-terms with the same type. The substitution of a for $\underline{n}$ in b, written $b\{\underline{n}/a\}$, is defined by induction over the structure of b as follows:*

$(a)\ X\{\underline{n}/a\} = X.$ $\qquad$ $(b)\ \underline{m}\{\underline{n}/a\} = \begin{cases} \underline{m}, & \text{if } m<n;\\ a, & \text{if } m=n;\\ \underline{m-1}, & \text{if } m>n.\end{cases}$

$(c)\ (c\ d)\{\underline{n}/a\} = c\{\underline{n}/a\}\ d\{\underline{n}/a\}$ $\qquad$ $(d)\ (\lambda_A.c)\{\underline{n}/a\} = \lambda_A.c\{\underline{n+1}/a^{+}\}$

This definition of a (higher-order) substitution is specific for β-reduction: in fact, in item (b), if $m>n$ then $\underline{m}\{\underline{n}/a\}$ is equal to $\underline{m-1}$ because this substitution assumes that a λ disappeared after an application of a β-reduction and, hence all the constants of the current term need to be decremented by one because now they are under the scope of one less abstractor. When the substitution $\{\underline{n}/a\}$ is propagated inside an abstraction, the term a^{+} is generated. It is called the *lift* of a and is given by the following definition:

Definition 16. *Let $a \in \Lambda_{dB}(\mathcal{X})$, $i \geq 0$. The term a^+, called the lift of a, is defined by $a^+ = a^{+0}$, where a^{+i} is inductively defined by:*

(a) $X^{+i} = X$, *for* $X \in \mathcal{X}$.

(b) $\underline{n}^{+i} = \begin{cases} \underline{n+1}, & \text{if } n > i; \\ \underline{n}, & \text{if } n \leq i. \end{cases}$

(c) $(a\ b)^{+i} = a^{+i}\ b^{+i}$.

(d) $(\lambda_A.a)^{+i} = \lambda_A.a^{+(i+1)}$.

Notice that we have defined the β-reduction as well as the notion of higher-order substitution for β-reduction without the decoration with types and contexts. Nevertheless, since we are interested in higher-order unification, it is important to keep in mind how this information is manipulated. The decorated version of the β-reduction is given by:

$$(\lambda_A.a_B^{A\cdot\Gamma})\ b_A^\Gamma \to a_B^{A\cdot\Gamma}\{\underline{1}_A^{A\cdot\Gamma}/b_A^\Gamma\} \qquad (\beta)$$

This definition says that each free occurrence of $\underline{1}_A^{A\cdot\Gamma}$ in $a_B^{A\cdot\Gamma}$ must be replaced by b_A^Γ, and that is the reason why $\underline{1}_A^{A\cdot\Gamma}$ and $a_B^{A\cdot\Gamma}$ must have the same context. One important point about a substitution generated by β-reductions is that it always has the form $\{\underline{n}/b\}$ where $\underline{n}$ and b are terms of the same type but with different contexts. In fact, the free occurrences of the de Bruijn index $\underline{n}$ in the term a are in the scope of the abstractor that will be removed after the application of the (β), but the term b is not in the scope of this abstractor. In what follows, we present the decorated version of definitions 16 and 15.

Remark 17. Let Δ be a context and $A, B, A_1, \ldots, A_n$ be types. The decorated version of Definition 15 is given in what follows. Assuming that the considered substitution was originated by a β-redex whose abstractor was of type A_1, we have that:

(a) $X_A^{A_n\cdot\ldots\cdot A_1\cdot\Delta}\{\underline{n}_B^{A_n\cdot\ldots\cdot A_1\cdot\Delta}/a_B^{A_n\cdot\ldots\cdot A_2\Delta}\} = X_A^{A_n\cdot\ldots\cdot A_2\Delta}$, i.e., meta-variables are not affected by the substitutions generated by β-reduction, but the context of the resulting term is given by the context of the term a given in the substitution.

(b) It is divided in three sub-cases:

- If $m < n$ then $\underline{m}$ represents a bound de Bruijn index and must remain unchanged:

$$\underline{m}_A^{A_n\cdot\ldots\cdot A_1\cdot\Delta}\{\underline{n}_B^{A_n\cdot\ldots\cdot A_1\cdot\Delta}/a_B^{A_n\cdot\ldots\cdot A_2\cdot\Delta}\} = \underline{m}_A^{A_n\cdot\ldots\cdot A_2\cdot\Delta}.$$

- If $m = n$ then

$$\underline{n}_A^{A_n\cdot\ldots\cdot A_1\cdot\Delta}\{\underline{n}_A^{A_n\cdot\ldots\cdot A_1\cdot\Delta}/a_A^{A_n\cdot\ldots\cdot A_2\cdot\Delta}\} = a_A^{A_n\cdot\ldots\cdot A_2\cdot\Delta}.$$

- If $m > n$ then the de Bruijn index $\underline{m}$ represents a constant whose scope contains one less abstractor (the one that originated the β-reduction was eliminated) and, then the first element of the context (whose type is exactly the type of the eliminated abstractor) is removed:

$$\underline{m}_A^{A_n\cdot\ldots\cdot A_1\cdot\Delta}\{\underline{n}_B^{A_n\cdot\ldots\cdot A_1\cdot\Delta}/a_B^{A_n\cdot\ldots\cdot A_2\cdot\Delta}\} = \underline{m-1}_B^{A_n\cdot\ldots\cdot A_2\cdot\Delta}.$$

(c) Trivial

(d) After propagating a substitution inside an abstractor of a term, the index $\underline{n}$ that defines the substitution and the term in the substitution as well as their contexts need to be updated:
$((\lambda_B.b_A^{B\cdot A_n\cdot\ldots\cdot A_1\cdot\Delta})_{B\to A}^{A_n\cdot\ldots\cdot A_1\cdot\Delta}\{\underline{n}_{A_1}^{A_n\cdot\ldots\cdot A_1\cdot\Delta}/a_{A_1}^{A_n\cdot\ldots\cdot A_2\cdot\Delta}\})_{B\to A}^{A_n\cdot\ldots\cdot A_2\cdot\Delta} =$
$(\lambda_B.(b_A^{B\cdot A_n\cdot\ldots\cdot A_1\cdot\Delta}\{\underline{n+1}_{A_1}^{B\cdot A_n\cdot\ldots\cdot A_1\cdot\Delta}/(a^+)_{A_1}^{B\cdot A_n\cdot\ldots\cdot A_2\cdot\Delta}\}))_{B\to A}^{A_n\cdot\ldots\cdot A_2\cdot\Delta}$.
In this way, the lift increases the context of the terms in the substitutions with the type of the abstractor that binds them.

The lift of Definition 16 is motivated by item (d) above. In fact, the lift of a term is necessary only when a substitution is propagated inside an abstraction whose type is essential to determine the resulting context. In the following, we assume that B is the type of the abstraction that originated the lift. The decorated version of the lift is given by:

(a) For all $i \geq 0$, $((X_A^{A_1\cdot\ldots\cdot A_i\cdot\Delta})^{+i})_A^{A_1\cdot\ldots\cdot A_i\cdot B\cdot\Delta} = X_A^{A_1\cdot\ldots\cdot A_i\cdot B\cdot\Delta}$.

(b) We analyze each case separately:

- If $n > i$ then the index $\underline{n}$ represents a constant that is in the scope of i abstractors and, that now was inserted (by the substitution) in the scope of a new abstractor of type B. Therefore it need to be updated and, its context must contain the type information concerning this new abstractor:

$$((\underline{n}_{A_n}^{A_1\cdot\ldots\cdot A_i\cdot\Delta})^{+i})_{A_n}^{A_1\cdot\ldots\cdot A_i\cdot B\cdot\Delta} = \underline{n+1}_{A_n}^{A_1\cdot\ldots\cdot A_i\cdot B\cdot\Delta}$$

- If $n \leq i$ then $\underline{n}$ represents a bound variable and must remain unchanged but the resulting context depends on the lift:

$$((\underline{n}_{A_n}^{A_1\cdot\ldots\cdot A_i\cdot\Delta})^{+i})_{A_n}^{A_1\cdot\ldots\cdot A_i\cdot B\cdot\Delta} = \underline{n}_{A_n}^{A_1\cdot\ldots\cdot A_i\cdot B\cdot\Delta}$$

(c) Trivial.

(d) To propagate a lift inside an abstraction, it is necessary to include the type of the abstraction that originated the lift:
$(((\lambda_A.a_C^{A\cdot A_1\cdot\ldots\cdot A_i\cdot\Delta})_{A\to C}^{A_1\cdot\ldots\cdot A_i\cdot\Delta})^{+i})_{A\to C}^{A_1\cdot\ldots\cdot A_i\cdot B\cdot\Delta} =$
$(\lambda_A.(a^{i+1})_C^{A\cdot A_1\cdot\ldots\cdot A_i\cdot B\cdot\Delta})_{A\to C}^{A_1\cdot\ldots\cdot A_i\cdot B\cdot\Delta}$.

Whenever it is possible we avoid the decorated notation for the sake of clarity. The η-reduction for the λ-calculus in de Bruijn's notation is defined as follows:

$$\lambda_A.a\ \underline{1} \to b \text{ if } a = b^+ \qquad (\eta)$$

and its version decorated with types and contexts is given by:

$$\lambda_A.a_B^{A\cdot\Gamma}\ \underline{1}_A^{A\cdot\Gamma} \to b_B^{\Gamma} \text{ se } a_B^{A\cdot\Gamma} = ((b_B^{\Gamma})^+)_B^{A\cdot\Gamma} \qquad (\eta)$$

The above definition of η-reduction tries to capture the operational semantics of the η-reduction of the λ-calculus with names, but in fact it fails because it

does not show how to construct the term b from a. However, implementations of the η-reduction based on detection of occurrences of the index $\underline{1}$ in a are adequate [AMK05].

As mentioned before, the separation of the free variables (meta-variables) on one side and the bound variables and constants (de Bruijn indexes) on the other side allow us to distinguish between the substitutions generated by β-reductions from the ones generated by the unification procedure. The next definition formalizes the notion of substitution generated by the unification procedure, i.e., substitutions for meta-variables:

Definition 18. *Let θ be a valuation (i.e., a function) from $\mathcal{X}$ to $\Lambda_{dB}(\mathcal{X})$. The* substitution θ' *extending the valuation θ is defined by:*
$(a)\ X\theta' = X\theta$ $\quad$ $(b)\ \underline{n}\theta' = \underline{n}$
$(c)\ (a\ b)\theta' = a\theta'\ \ b\theta'$ $\quad$ $(d)\ (\lambda_A.a)\theta' = \lambda_A.(a\theta'^{+})$
where $\theta'^{+} := \{X_1^+/a_1^+, \ldots, X_n^+/a_n^+\}$ *when* $\theta' = \{X_1/a_1, \ldots, X_n/a_n\}$.

The main difference between the substitution generated by the β-reduction and the one generated by the unification procedure is that the latter always replaces a meta-variable for a term, say X/a, where X and a are λ-terms with the same type and context. Here again, contexts need to be updated when propagated over abstractions: for instance, the decorated version of item (d) is given by $(\lambda_B.c_C^{B\cdot\Delta})\{X_A^\Gamma/a_A^\Gamma\} = \lambda_B.(c_C^{B\cdot\Delta}\{X_A^{B\cdot\Gamma}/(a^+)_A^{B\cdot\Gamma}\})$.

The next example clarifies the process of propagating substitutions and the notion of lifting.

Example 19. Let $\Gamma = (A \to A) \to A$ be a context and, X be a meta-variable of type $(A \to A) \to A$ in context Γ. In this example, we show how the substitution $\{X^\Gamma_{(A\to A)\to A}/\underline{1}^\Gamma_{(A\to A)\to A}\}$ can be propagated over the λ-term

$$\lambda_{A\to A}.(X\ \lambda_A.(X\ \underline{2}))$$

that has type $(A \to A) \to A$ in context Γ. Due to lack of space, we only decorate the sub-terms that are relevant while propagating the substitution:
$(\lambda_{A\to A}.(X\ \lambda_A.(X\ \underline{2})))^\Gamma_{(A\to A)\to A}\{X^\Gamma_{(A\to A)\to A}/\underline{1}^\Gamma_{(A\to A)\to A}\} =$
$\lambda_{A\to A}.(X\ \lambda_A.(X\ \underline{2}))^{A\to A\cdot\Gamma}_A\{X^{A\to A\cdot\Gamma}_{(A\to A)\to A}/\underline{2}^{A\to A\cdot\Gamma}_{(A\to A)\to A}\} =$
$\lambda_{A\to A}.(\underline{2}^{A\to A\cdot\Gamma}_{(A\to A)\to A}\ (\lambda_A.(X\ \underline{2}))^{A\to A\cdot\Gamma}_{A\to A}\{X^{A\to A\cdot\Gamma}_{(A\to A)\to A}/\underline{2}^{A\to A\cdot\Gamma}_{(A\to A)\to A}\}) =$
$\lambda_{A\to A}.(\underline{2}^{A\to A\cdot\Gamma}_{(A\to A)\to A}\ \lambda_A.(X\ \underline{2})^{A\cdot A\to A\cdot\Gamma}_A\{X^{A\cdot A\to A\cdot\Gamma}_{(A\to A)\to A}/\underline{3}^{A\cdot A\to A\cdot\Gamma}_{(A\to A)\to A}\}) =$
$\lambda_{A\to A}.(\underline{2}^{A\to A\cdot\Gamma}_{(A\to A)\to A}\ \lambda_A.(\underline{3}^{A\cdot A\to A\cdot\Gamma}_{(A\to A)\to A}\ \underline{2}^{A\cdot A\to A\cdot\Gamma}_{A\to A}))$.

In this example, the lift was used twice: once in the second line (top-down) and once in the fourth line.

In the following we define the *updating functions* that are used in the definition of η-long normal forms for λ-terms in de Bruijn's notation.

Definition 20. *The* updating functions $U^i_k : \Lambda_{dB}(\mathcal{X}) \to \Lambda_{dB}(\mathcal{X})$*, for $k \geq 0$ and $i \geq 1$ are defined inductively by:*
$(a)\ U^i_k(X) = X,$ *for* $X \in \mathcal{X}$ $\quad$ $(b)\ U^i_k(a\ b) = U^i_k(a)\ U^i_k(b)$
$(c)\ U^i_k(\lambda_A.a) = \lambda_A.U^i_{k+1}(a)$ $\quad$ $(d)\ U^i_k(\underline{n}) = \begin{cases} \underline{n+i-1}, & \text{if } n > k \\ \underline{n}, & \text{if } n \leq k \end{cases}$

In the λ-calculus, η-long forms play an important role. The following definition was adapted from [DHK00]:

Definition 21 (η-long nf). *Let $a \in \Lambda_{dB}(\mathcal{X})$ be a λ-term in de Bruijn's notation of type $A_1 \to \ldots \to A_m \to B$ (B atomic) in context Γ and in β-nf. The* η-long normal form *(or η-lnf for short) a' of a, is inductively defined by:*

- *if $a = \lambda_A.b$ then $a' = \lambda_A.b'$.*
- *if $a = \underline{n}\ b_1 \ldots b_q$, with $q \geq 0$, then $a' = \lambda_{A_1} \ldots \lambda_{A_m}.\underline{n+m}\ c_1 \ldots c_q\ \underline{m}' \ldots \underline{1}'$, where $c_1, \ldots, c_q$ are the η-lnf of the β-nf of $U_0^{m+1}(b_1), \ldots, U_0^{m+1}(b_q)$, resp.*
- *if $a = X\ b_1 \ldots b_q$, with $q \geq 0$, then $a' = \lambda_{A_1} \ldots \lambda_{A_m}.X\ c_1 \ldots c_q\ \underline{m}' \ldots \underline{1}'$, where $c_1, \ldots, c_q$ are the η-lnf of the β-nf of $U_0^{m+1}(b_1), \ldots, U_0^{m+1}(b_q)$, resp.*

The next definition is needed to prove that the definition of η-lnf is well founded.

Definition 22. *The size $|a|$ of a λ-term $a \in \Lambda_{dB}(\mathcal{X})$ is inductively defined by:*

- *if $a = \underline{n}$ or $a = X$ then $|a| = 1$;*
- *if $a = b\ c$ then $|a| = 1 + |b| + |c|$;*
- *if $a = \lambda_A.b$ then $|a| = 1 + |b|$.*

Proposition 23. *The definition of η-lnf for λ-terms in de Bruijn's notation is well founded.*

Proof. The proof is by induction based on the lexicographic order on the triple consisting of the number of occurrences of meta-variables, the size of the λ-term and the size of its type. The size of a type is defined as usual: if A is atomic then $|A| = 1$ and if B and C are types then $|B \to C| = max(1 + |B|, |C|)$.

In the case $a = \lambda_A.b$ we have that the number of meta-variables remain unchanged and the size of the term decreases. When $a = \underline{n}\ b_1 \ldots b_q$ and $q = 0$ the number of meta-variables and the size of the term remain unchanged but the size of the type decreases. If $q \neq 0$ then the number of meta-variables remain unchanged and the size of the term decreases. When $a = X\ b_1 \ldots b_q$ the number of meta-variables decreases. □

Example 24. Consider the following type judgment $A \to A \cdot nil \vdash \underline{1} : A \to A$. The η-lnf of the de Bruijn index $\underline{1}$ in this type judgment, in a first step, corresponds to the η-lnf of $A \to A \cdot nil \vdash \lambda_A.\underline{2}\ \underline{1}' : A \to A$. But the η-lnf of a de Bruijn index of an atomic type is the index itself, and therefore, the η-lnf of the original term is given by $A \to A \cdot nil \vdash \lambda_A.\underline{2}\ \underline{1} : A \to A$.

Example 25. A more interesting case is to calculate the η-lnf of $(A \to A) \to A \cdot nil \vdash \underline{1} : (A \to A) \to A$. According to the definition it corresponds to the η-lnf of $(A \to A) \to A \cdot nil \vdash \lambda_{A \to A}.\underline{2}\ \underline{1}' : (A \to A) \to A$. Now the problem is reduced to calculating the η-lnf of the λ-term $\underline{1}'$ that has type $A \to A$ in context $A \to A \cdot (A \to A) \to A \cdot nil$. Following the previous example, we have that the η-lnf of $A \to A \cdot (A \to A) \to A \cdot nil \vdash \underline{1}' : A \to A$ is given by $A \to A \cdot (A \to A) \to A \cdot nil \vdash \lambda_A.\underline{2}\ \underline{1} : A \to A$. Therefore, we have that the η-lnf of the original term is given by $(A \to A) \to A \cdot nil \vdash \lambda_{A \to A}.\underline{2}\ \lambda_A.\underline{2}\ \underline{1} : (A \to A) \to A$.

3.2 Huet's algorithm in de Bruijn's notation

The definitions of normal forms, flexible and rigid terms and unification problem for de Bruijn's notation is a straightforward adaptation from those given in subsection 2.2.

In the next subsections we present the procedures SIMPL and MATCH of Huet's algorithm in de Bruijn's notation using the unification tree notation.

The procedure SIMPL It receives as argument a unification problem P_α containing at least one rigid-rigid equation (otherwise it is already a simplified problem) and returns either a terminal (Success or Fail) or an equivalent (simplified) unification problem, written $\overline{P_\alpha}$, containing at least one flexible-rigid equation. In the following we give a description of SIMPL.

Procedure SIMPL

`INPUT`: A unification problem P_α with at least one rigid-rigid equation.

`OUTPUT`: Either a terminal (Success or a Fail) or an equivalent unification problem $\overline{P_\alpha}$ without rigid-rigid equations and containing at least one flexible-rigid equation.

`WHILE` there exists a rigid-rigid equation in P_α, say:

$$\lambda_{A_1} \dots \lambda_{A_n}.h_1\ e_1^1 \dots e_{p_1}^1 =^? \lambda_{A_1} \dots \lambda_{A_n}.h_2\ e_1^2 \dots e_{p_2}^2 \wedge P' \tag{5}$$

where $n, p_1, p_2 \geq 0$ and h_1 and h_2 are de Bruijn indexes `DO`

If h_1 and h_2 are different de Bruijn indexes then stop and report a failure status. Otherwise, replace the equation (5) (in which $p_1 = p_2$ because the terms have the same type) by the conjunction

$$\lambda_{A_1} \dots \lambda_{A_n}.e_1^1 =^? \lambda_{A_1} \dots \lambda_{A_n}.e_1^2 \wedge \dots \wedge \lambda_{A_1} \dots \lambda_{A_n}.e_{p_1}^1 =^? \lambda_{A_1} \dots \lambda_{A_n}.e_{p_1}^2$$

in P_α and call $\overline{P_\alpha}$ the resulting problem.
`DONE`.
`IF` there exists a flexible-rigid equation in $\overline{P_\alpha}$ `THEN` return $\overline{P_\alpha}$ `ELSE` stop and report a success status.

The Procedure MATCH It takes a flexible-rigid equation as argument and returns a finite set of substitutions called Σ. As explained for the notation with names, the procedure MATCH is based on the *imitation* and *projection* rules detailed in the following.

The Imitation Rule. Consider the following flexible-rigid equation well typed in context Γ:

$$\lambda_{A_1} \dots \lambda_{A_n}.X\ e_1^1 \dots e_{p_1}^1 =^? \lambda_{A_1} \dots \lambda_{A_n}.\underline{h}\ e_1^2 \dots e_{p_2}^2 \tag{6}$$

where:

- $n, p_1, p_2 \geq 0$
- X is a meta-variable of type $B_1 \to \ldots \to B_{p_1} \to A$ (A atomic)
- $\underline{h}$ is a de Bruijn index of type $C_1 \to \ldots \to C_{p_2} \to A$ (A atomic)
- if $p_1 \neq 0$ then e_i^1 is a λ-term in η-lnf of type B_i for all $1 \leq i \leq p_1$
- if $p_2 \neq 0$ then e_j^2 is a λ-term in η-lnf of type C_j for all $1 \leq j \leq p_2$

In order to avoid variable capture, an imitation substitution is generated only if $\underline{h}$ is a constant, i.e., $h > n$. In this case, the imitation substitution is given by:

$$X/\lambda_{B_1} \ldots \lambda_{B_{p_1}}.\underline{p_1 + h - n}\ (X_1\ \underline{p_1} \ldots \underline{1}) \ldots (X_{p_2}\ \underline{p_1} \ldots \underline{1})$$

where X_i is a fresh meta-variable of type $B_1 \to \ldots \to B_{p_1} \to C_i$ in context Γ, for all $1 \leq i \leq p_2$.

The Projection Rule. For each argument of X (in equation (6)) that have the same target type as X a projection is generated. In this way, if e_i^1 has a type of the form $B_i = D_1 \to \ldots \to D_q \to A$, for some $1 \leq i \leq p_1$, then the generated projection substitution is given by:

$$X/\lambda_{B_1} \ldots \lambda_{B_{p_1}}.\underline{p_1 - i + 1}\ (H_1\ \underline{p_1} \ldots \underline{1}) \ldots (H_q\ \underline{p_1} \ldots \underline{1})$$

where H_j is a fresh meta-variable of type $B_1 \to \ldots \to B_{p_1} \to D_j$ for all $1 \leq j \leq q$.

In the following we give an algorithmic description of the procedure MATCH.

Procedure MATCH

`INPUT`: A flexible-rigid equation eq.
`OUTPUT`: A set Σ of substitutions for the head of the flexible term.

1. Apply the imitation and the projection rules to eq non-deterministically and call Σ the set of generated substitutions.

The Main Procedure The main procedure of Huet's algorithm non-deterministically and successively calls the procedures SIMPL and MATCH.

Main Procedure

`INPUT`: A unification problem P_ϵ.
`OUTPUT`: A success status if the original problem is unifiable or a failure status if the original problem is not unifiable. The algorithm may not terminate in the latter case.

1. If $P_{i_1 \ldots i_k}$ contains a rigid-rigid equation then apply SIMPL and go to the next step, else if it contains a flexible-rigid equation then rename $P_{i_1 \ldots i_k}$ to $\overline{P}_{i_1 \ldots i_k}$ and go to the next step, else go to step 4.
2. Let eq be a flexible-rigid equation in $\overline{P}_{i_1 \ldots i_k}$. Apply MATCH to eq and call $\Sigma_{i_1 \ldots i_k}$ the generated set of substitutions and go to step 3.

3. If $\Sigma_{i_1 \ldots i_k}$ is the empty set then stop and report a failure status, else let $\Sigma_{i_1 \ldots i_k} = \{\sigma_{i_1 \ldots i_k 1}, \ldots, \sigma_{i_1 \ldots i_k r}\}$ where $r > 0$ and, for each substitution $\sigma_{i_1 \ldots i_k j} \in \Sigma_{i_1 \ldots i_k}$ call $P_{i_1 \ldots i_k j} := P_{i_1 \ldots i_k} \sigma_{i_1 \ldots i_k j}$ the new unification problem and go to step 1.
4. Stop and report a success status. The corresponding solution assuming that the current node is at position $i_1 \ldots i_k$ is given by the composition: $\sigma_{i_1} \sigma_{i_1 i_2} \ldots \sigma_{i_1 i_2 \ldots i_{k-1}} \sigma_{i_1 i_2 \ldots i_k}$

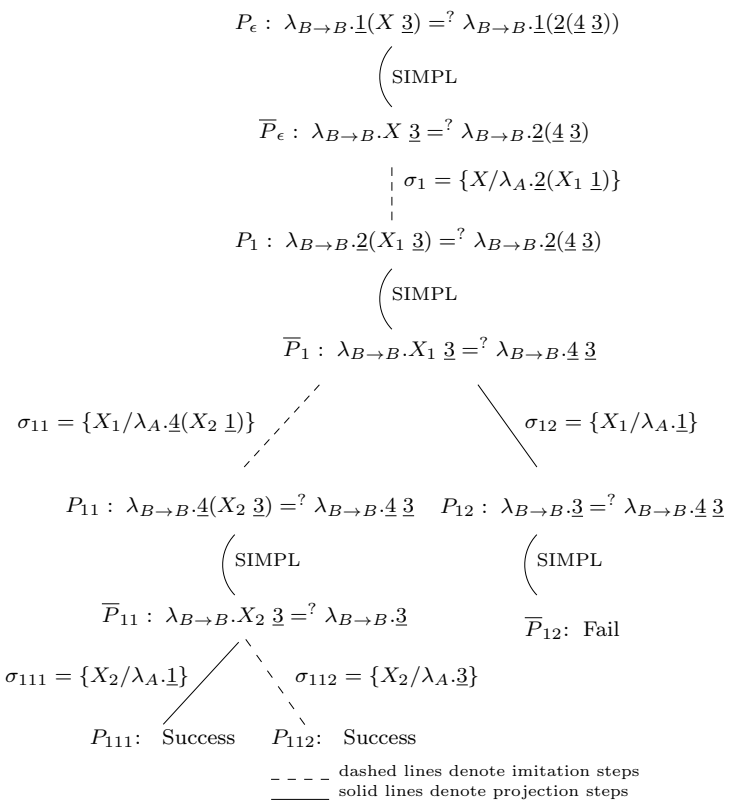


Fig. 4. Unification tree example.

Example 26. Consider the unification problem given in Example 8. First of all, we need to rewrite its terms in de Bruijn's notation. To do so, we choose the referential $u : A \to B, w : A, v : A \to A$. This referential corresponds to the context $\Gamma = A \to B \cdot A \cdot A \to A \cdot nil$. As explained in Section 3.1 this corresponds to considering the terms of the equation

$$\lambda_{y:B\to B}.y\ (X\ w) =^? \lambda_{x:B\to B}.x\ (u\ (v\ w))$$

under the scope of the abstractors $\lambda_{v:A\to A}\lambda_{w:A}\lambda_{u:A\to B}$ and we get:

$$\lambda_{B\to B}.\underline{1}(X\ \underline{3}) =^{?} \lambda_{B\to B}.\underline{1}(\underline{2}(\underline{4}\ \underline{3}))$$

is well typed in context Γ.

Figure 4 shows the unification tree generated for this problem. The solutions are given by the compositions of the substitutions through a path whose leaf is a success node, i.e., this node corresponds to a problem containing at most a finite number of flexible-flexible equations. Note that, after composing, the terms need to be normalized. For instance, one can compute the composition $\sigma_1\sigma_{11}\sigma_{112}$ by first applying the usual composition of substitutions:

$$\{X/\lambda_A.\underline{2}((\lambda_A.\underline{5}((\lambda_A.\underline{1})\ \underline{1}))\ \underline{1}), X_1/\lambda_A.\underline{4}((\lambda_A.\underline{1})\ \underline{1}), X_2/\lambda_A.\underline{1}\}$$

and then applying β-reduction:

$$\{X/\lambda_A.\underline{2}(\underline{4}\ \underline{1}), X_1/\lambda_A.\underline{4}\ \underline{1}, X_2/\lambda_A.\underline{1}\}.$$

In the same way, one computes the substitution

$$\sigma_1\sigma_{11}\sigma_{112} = \{X/\lambda_A.\underline{2}(\underline{4}\ \underline{3}), X_1/\lambda_A.\underline{4}\ \underline{3}, X_2/\lambda_A.\underline{3}\}.$$

The solutions to the original problem are given by the substitutions for the meta-variables that appear in it: $X/\lambda_A.\underline{2}(\underline{4}\ \underline{3})$ and $X/\lambda_A.\underline{2}(\underline{4}\ \underline{1})$.

4 The Relation between HOU in the λ-calculus and in the $\lambda\sigma$-calculus

In this section, we relate, HOU à la Huet and HOU in the $\lambda\sigma$-calculus. In [DHK00], Dowek, Hardin and Kirchner prove that a unification problem in the simply typed λ-calculus has a solution if and only if its precooked image has a solution. In this paper we go a step further and show that, for each subproblem P_α of a given problem P, there exists a subproblem P^*_B of the precooked image of P that preserves solutions in the following sense: if the substitution σ is a solution to P_α then the grafting σ_F is a solution to P^*_B. We start with a brief presentation of the $\lambda\sigma$-calculus.

4.1 The $\lambda\sigma$-calculus

The λ-calculus is based on a notion of substitution that belongs to a meta-language. Such a notion is necessary because the substitution process adopts renaming of bound variables in order to avoid variable capture. A natural solution to define a substitution which belongs to the language itself is to extend the language of the λ-calculus by incorporating explicit operators for the substitution. The first mechanism that "explicited" the substitution operation was the $\lambda\sigma$-calculus [ACCL91] that we briefly present in the following.

Definition 27. *The syntax of the simply typed $\lambda\sigma$-calculus is given by:*

Types	$A ::= K \mid A \to A$	*where* $K \in \mathbb{T}$
Contexts	$\Gamma ::= nil \mid A \cdot \Gamma$	
Terms	$a ::= \underline{1} \mid X \mid a\ a \mid \lambda_A.a \mid a[s]$	*where* $X \in \mathcal{X}$
Substitutions	$s ::= id \mid \uparrow \mid a \cdot s \mid s \circ s$	

The set of well typed $\lambda\sigma$-terms with meta-variables is denoted by $\Lambda_{\lambda\sigma}(\mathcal{X})$. The typing rules for the $\lambda\sigma$-calculus are as follows:

$$(var)\ \frac{}{A \cdot \Gamma \vdash \underline{1} : A} \qquad (lambda)\ \frac{A \cdot \Gamma \vdash a : B}{\Gamma \vdash \lambda_A.a : A \to B}$$

$$(app)\ \frac{\Gamma \vdash a : A \to B \quad \Gamma \vdash b : A}{\Gamma \vdash (a\ b) : B} \qquad (clos)\ \frac{\Gamma \vdash s \triangleright \Gamma' \quad \Gamma' \vdash a : A}{\Gamma \vdash a[s] : A}$$

$$(id)\ \frac{}{\Gamma \vdash id \triangleright \Gamma} \qquad (shift)\ \frac{}{A \cdot \Gamma \vdash \uparrow \triangleright \Gamma}$$

$$(cons)\ \frac{\Gamma \vdash a : A \quad \Gamma \vdash s \triangleright \Gamma'}{\Gamma \vdash a \cdot s \triangleright A \cdot \Gamma'} \qquad (comp)\ \frac{\Gamma \vdash s'' \triangleright \Gamma'' \quad \Gamma'' \vdash s' \triangleright \Gamma'}{\Gamma \vdash s' \circ s'' \triangleright \Gamma'}$$

In addition, to each meta-variable X we associate a unique type T_X and a unique context Γ_X. We add the following type rule for meta-variables:

$$(meta)\ \frac{}{\Gamma_X \vdash X : T_X}$$

In contrast to the (*meta*) rule of the simply typed λ-calculus (in de Bruijn's notation), the (*meta*) rule for the $\lambda\sigma$-calculus shows that the types of $\lambda\sigma$-terms are not independent from the contexts. This is necessary because unification in the $\lambda\sigma$-calculus uses grafting instead of substitution; and we would like grafting and typing to be compatible in the $\lambda\sigma$-calculus. This restriction over meta-variables avoids, for example, the replacement of the two occurrences of X in the $\lambda\sigma$-term $(X\ \lambda_A.X)$ by the same $\lambda\sigma$-term (see Remark 34 for further details).

The rewriting rules of the $\lambda\sigma$-calculus are given in Table 1.

In this calculus, when a substitution s is applied to a term a we internalize this as $a[s]$. Simultaneous substitutions are represented as lists of terms with the usual operator *cons* (written as "$\cdot$") and an operator for the empty list (written *id* which represents the identity substitution) and the operator $\uparrow$ which represents the infinite substitution $\underline{2} \cdot \underline{3} \cdot \ldots$. Although the $\lambda\sigma$-calculus codifies the de Bruijn index $\underline{n}$ as $\underline{1}[\uparrow^{n-1}]$, for the sake of clarity, we will follow [DHK00] in not adopting such a codification.

The notion of normal form for $\lambda\sigma$-expressions is given in what follows:

Proposition 28 ([Río93]). *Any $\lambda\sigma$-term in* normal form *is of one of the following forms:*

1. *$\lambda_A.a$, where a is in normal form.*

(Beta)	$(\lambda.a)\ b$	$\longrightarrow a[b \cdot id]$
(App)	$(a\ b)[s]$	$\longrightarrow a[s]\ b[s]$
(Abs)	$(\lambda.a)[s]$	$\longrightarrow \lambda.a[\underline{1} \cdot (s \circ \uparrow)]$
(Clos)	$(a[s])[t]$	$\longrightarrow a[s \circ t]$
(VarCons)	$\underline{1}[a \cdot s]$	$\longrightarrow a$
(Id)	$a[id]$	$\longrightarrow a$
(Assoc)	$(s \circ t) \circ u$	$\longrightarrow s \circ (t \circ u)$
(Map)	$(a \cdot s) \circ t$	$\longrightarrow a[t] \cdot (s \circ t)$
(IdL)	$id \circ s$	$\longrightarrow s$
(IdR)	$s \circ id$	$\longrightarrow s$
(ShiftCons)	$\uparrow \circ (a \cdot s)$	$\longrightarrow s$
(VarShift)	$\underline{1} \cdot \uparrow$	$\longrightarrow id$
(SCons)	$\underline{1}[s] \cdot (\uparrow \circ s)$	$\longrightarrow s$
(Eta)	$\lambda.a\ \underline{1}$	$\longrightarrow b \quad \text{if} \quad a =_\sigma b[\uparrow]$

Table 1. The $\lambda\sigma$-rewriting system with η-conversion

2. *$a\ b_1 \ldots b_q$, where a and b_i are in normal form and a is either $\underline{1}$, $\underline{1}[\uparrow^n]$, X or $X[s]$ where s is a substitution term in normal form and different from id.*
3. *$a_1 \cdot \ldots \cdot a_p \cdot \uparrow^n$, where $a_1, \ldots, a_p$ are in $\lambda\sigma$-terms in normal form and $a_p \neq \underline{n}$.*

$\lambda\sigma$-terms in η-lnf are given in what follows:

Definition 29 (η-lnf [DHK00]). *Let a be a $\lambda\sigma$-term of type $A_1 \to \ldots \to A_n \to B$ in context Γ and in $\lambda\sigma$-nf. The* η-lnf *of a, written as a', is defined by:*

1. *If $a = \lambda_A.b$ then $a' = \lambda_A.b'$.*
2. *If $a = \underline{k}\ b_1 \ldots b_q$ then $a' = \lambda_{A_1} \ldots \lambda_{A_n}.\underline{k+n}\ c_1 \ldots c_q\ \underline{n}' \ldots \underline{1}'$, where c_i is the η-lnf of the normal form of $b_i[\uparrow^n]$.*
3. *If $a = X[s]\ b_1 \ldots b_q$ then $a' = \lambda_{A_1} \ldots \lambda_{A_n}.X[s']\ c_1 \ldots c_q\ \underline{n}' \ldots \underline{1}'$, where c_i is the η-lnf of the normal form of $b_i[\uparrow^n]$ and if $s = d_1 \cdot \ldots \cdot d_r \cdot \uparrow^k$ then $s' = e_1 \cdot \ldots \cdot e_r \cdot \uparrow^{k+n}$ where e_i is the η-lnf form of $d_i[\uparrow^n]$.*

Remark 30. Definition 29 is shown to be well founded in [DHK00]. In addition, we should note that "in the $\lambda\sigma$-calculus, the reduction of an η-redex may create a σ-redex. For instance, the term $X[\lambda_A.(\underline{2}\ \underline{1}) \cdot \uparrow]$ reduces to $X[\underline{1} \cdot \uparrow]$ then to $X[id]$ then to X. Thus to compute the η-lnf we need to reduce all the redexes (including the η ones) before expanding the term."

4.2 Unification in the $\lambda\sigma$-calculus

A unification problem in the $\lambda\sigma$-calculus is written as a disjunction of existentially quantified conjunctions of the form $\bigvee_{j \in J} \exists \overrightarrow{w}_j \bigwedge_{i \in I_j} a_i =^?_{\lambda\sigma} b_i$ where a_i and

b_i are $\lambda\sigma$-terms of the same type and, all the terms of the problem are typed in the same context. This is necessary because the same meta-variable may appear in different equations and they are supposed to be replaced by the same term. If $|J| = 1$ then the unification problem is called a *unification system.*

In the unification method over the $\lambda\sigma$-calculus, the solutions are given by the solved forms which are defined as follows:

Definition 31 ([DHK00]). *A unification system P is in* $\lambda\sigma$-solved form *if it is a conjunction of nontrivial equations of the following forms:*

- **Solved***: $X =^{?}_{\lambda\sigma} a$, where the meta-variable X does not appear anywhere else in P and a is in η-lnf. Such an equation is said to be* solved *in P and the variable X is also said to be solved.*
- **Flexible-flexible***: $X[a_1 \cdot \ldots \cdot a_p \cdot \uparrow^n] =^{?}_{\lambda\sigma} Y[b_1 \cdot \ldots \cdot b_q \cdot \uparrow^m]$, where $X[a_1 \cdot \ldots \cdot a_p \cdot \uparrow^n]$ and $Y[b_1 \cdot \ldots \cdot b_q \cdot \uparrow^m]$ are in η-lnf and the equation is not solved.*

Since we are interested in relating a unification algorithm in the simply typed λ-calculus and one in the $\lambda\sigma$-calculus, it is important to know how to translate unification problems from one language to the other. The process of translating a unification problem P from the simply typed λ-calculus to the language of the simply typed $\lambda\sigma$-calculus is done by the *precooking* translation defined as follows:

Definition 32 (Precooking [DHK00]). *Let $a \in \Lambda_{dB}(\mathcal{X})$ such that $\Gamma \vdash a : A$. To every meta-variable X of type U in the term a, we associate the type U and context Γ in $\lambda\sigma$-calculus. The* precooking *of a from $\Lambda_{dB}(\mathcal{X})$ to $\Lambda_{\lambda\sigma}(\mathcal{X})$, is defined by $a_F = F(a, 0)$ where $F(a, n)$ is defined by:*

(a) $F((\lambda_B.a), n) = \lambda_B(F(a, n+1))$ (b) $F(\underline{k}, n) = \underline{1}[\uparrow^{k-1}]$

(c) $F(a\ b, n) = F(a, n)\ F(b, n)$ (d) $F(X, n) = X[\uparrow^n]$

Note that the precooking defined above is injective and hence its inverse is well defined. This remark will be important when unification solutions in the language of the $\lambda\sigma$-calculus need to be translated back to the language of the simply typed λ-calculus. Of course, some $\lambda\sigma$-terms cannot be translated back to the language of the simply typed λ-calculus by the inverse of the precooking translation. In addition, the precooking is a type preserving function as stated by the next proposition:

Proposition 33 ([DHK00]). *If $\Gamma \vdash a : A$ in $\Lambda_{dB}(\mathcal{X})$, then $\Gamma \vdash a_F : A$ in the $\Lambda_{\lambda\sigma}(\mathcal{X})$.*

Now we are ready to point out the fundamental importance of the precooking translation and how it deals with the differences of the (*meta*) rules in the λ-calculus and in the $\lambda\sigma$-calculus.

Remark 34. In this remark, we want to emphasize an important difference between the (*meta*) rule of the λ-calculus in de Bruijn's notation and the one of the $\lambda\sigma$-calculus. In the λ-calculus in de Bruijn's notation the types of meta-variables

are independent from the contexts. In fact, we can type the same meta-variable in different levels of abstraction: for instance, for a given context Γ, the λ-term $X\ \lambda_A.(X\ \lambda_A.\underline{1})$ is well typed in Γ, where X is a meta-variable of type $(A \to A) \to A$. The type of this term can be deduced as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \cdot A \cdot (A \to A) \to A \cdot nil \vdash \underline{1} : A}\,(var)}{A \cdot (A \to A) \to A \cdot nil \vdash \lambda_A.\underline{1} : A \to A}\,(lambda)}{A \cdot (A \to A) \to A \cdot nil \vdash (X\ \lambda_A.\underline{1}) : A}\,(app)}{(A \to A) \to A \cdot nil \vdash \lambda_A.(X\ \lambda_A.\underline{1}) : A \to A}\,(lambda)}{(A \to A) \to A \cdot nil \vdash X\ \lambda_A.(X\ \lambda_A.\underline{1}) : A}\,(app)$$

where $\boxdot$ corresponds to

$$\dfrac{}{A \cdot (A \to A) \to A \cdot nil \vdash X : (A \to A) \to A}\,(meta)$$

and $\boxtimes$ corresponds to

$$\dfrac{}{(A \to A) \to A \cdot nil \vdash X : (A \to A) \to A}\,(meta)$$

Nevertheless, seen as a $\lambda\sigma$-term, $X\ \lambda_A.(X\ \lambda_A.\underline{1})$ is not well typed; although it is well-formed! In fact, in the type derivation above we need to use $(meta)$ twice for the meta-variable X under different contexts which is allowed in the λ-calculus but not in the $\lambda\sigma$-calculus. In the $\lambda\sigma$-calculus each meta-variable has a unique context and hence, we cannot type the same meta-variable at different levels of abstraction. This means that in the $\lambda\sigma$-calculus the type of meta-variables depends on the context. This seems to be a severe restriction but this is necessary because in the $\lambda\sigma$-calculus one has grafting instead of substitution and, for instance, the application of the grafting $\{X \mapsto \underline{1}\}$ to the $\lambda\sigma$-term $X\ \lambda_A.(X\ \lambda_A.\underline{1})$ leads to $\underline{1}\ \lambda_A.(\underline{1}\ \lambda_A.\underline{1})$ which is not correct due to variable capture. The precooking translation is the key idea to solve this problem. In fact, the term that corresponds to $X\ \lambda_A.(X\ \lambda_A.\underline{1})$ in the $\lambda\sigma$-calculus is its precooked version given by $X\ \lambda_A.(X[\uparrow]\ \lambda_A.\underline{1})$ and which is well typed in the $\lambda\sigma$-calculus:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \cdot A \cdot (A \to A) \to A \cdot nil \vdash \underline{1} : A}\,(var)}{A \cdot (A \to A) \to A \cdot nil \vdash \lambda_A.\underline{1} : A \to A}\,(lambda)}{A \cdot (A \to A) \to A \cdot nil \vdash (X[\uparrow]\ \lambda_A.\underline{1}) : A}\,(app)}{(A \to A) \to A \cdot nil \vdash \lambda_A.(X[\uparrow]\ \lambda_A.\underline{1}) : A \to A}\,(lambda)}{(A \to A) \to A \cdot nil \vdash X\ \lambda_A.(X[\uparrow]\ \lambda_A.\underline{1}) : A}\,(app)$$

where $\boxtimes$ corresponds to

$$\dfrac{}{(A \to A) \to A \cdot nil \vdash X : (A \to A) \to A}\,(meta)$$

and $\boxdot$ corresponds to

$$\dfrac{\boxplus \qquad \dfrac{}{(A \to A) \to A \cdot nil \vdash X : (A \to A) \to A}\,(meta)}{A \cdot (A \to A) \to A \cdot nil \vdash X[\uparrow] : (A \to A) \to A}\,(clos)$$

where $\boxplus$ corresponds to

$$\dfrac{}{A \cdot (A \to A) \to A \cdot nil \vdash\!\uparrow \rhd (A \to A) \to A \cdot nil}\,(shift)$$

This example shows that the precooking translation performs the adequate adjustments to $\lambda\sigma$-terms which allow the use of grafting instead of substitution.

The unification rules for the $\lambda\sigma$-calculus are given in Table 2 which is taken from [DHK00]. This set of rules is called **Unif** and is assumed to be applied in a "fair" way: this means that applications of **Exp-λ** (the rule that introduces fresh meta-variables with simpler types) are always followed by applications of **Replace** to avoid infinite applications of **Exp-λ**. In fact, since an application of **Exp-λ** adds a new flexible-flexible equation and does not change anything else in the current problem, it could be applied *ad infinitum*.

A *derivation tree* is a tree that represents an application of the $\lambda\sigma$-HOU method. Formally, it is a tree with a unification system labeling its nodes. Moreover, the arcs that link the nodes are labeled with the unification rules presented in Table 2. Disjunctions of unification systems are represented as "or" branches as usual in tree descriptions. Figure 5 gives an example of a derivation tree.

Example 35. Consider again the context $\Gamma = A \to B \cdot A \cdot A \to A \cdot nil$ and the unification problem

$$\lambda_{B\to B}.\underline{1}(X\ \underline{3}) =^{?} \lambda_{B\to B}.\underline{1}(\underline{2}(\underline{4}\ \underline{3}))$$

whose unification tree is given in Figure 4. Applying the precooking, we get:

$$\lambda_{B\to B}.\underline{1}(X[\uparrow]\ \underline{3}) =^{?}_{\lambda\sigma} \lambda_{B\to B}.\underline{1}(\underline{2}(\underline{4}\ \underline{3})) \tag{7}$$

which is well typed in context Γ. The derivation tree of this system is presented in Figures 5, 6, 7, 9 and 8. Note that this derivation tree and the unification tree of Figure 4 have a similar structure: both have exactly one fail node and two success nodes. The solutions to equation (7) are given by the graftings $\{X \mapsto \lambda_A.(\underline{2}(\underline{4}\ \underline{1}))\}$ and $\{X \mapsto \lambda_A.(\underline{2}(\underline{4}\ \underline{3}))\}$ that correspond, respectively, to the substitutions $\{X/\lambda_A.\underline{2}(\underline{4}\ \underline{1})\}$ and $\{X/\lambda_A.\underline{2}(\underline{4}\ \underline{3})\}$ given in Example 26.

4.3 A Structural Relation Between HOU in the λ-calculus and in the λσ-calculus

In this section, we establish a relation between HOU in the λ-calculus and in the $\lambda\sigma$-calculus that refines a result established by Dowek, Hardin and Kirchner in [DHK00].

We start with the definition of the *pseudo-precooking* that extends the usual notion of precooking by combining it with some unification rules.

Dec-λ	$\dfrac{P \wedge \lambda_A.e_1 =^?_{\lambda\sigma} \lambda_A.e_2}{P \wedge e_1 =^?_{\lambda\sigma} e_2}$
Dec-App	$\dfrac{P \wedge (\underline{n}\ e_1^1 \dots e_p^1) =^?_{\lambda\sigma} (\underline{n}\ e_1^2 \dots e_p^2)}{P \wedge e_1^1 =^?_{\lambda\sigma} e_1^2 \wedge \dots \wedge e_p^1 =^?_{\lambda\sigma} e_p^2}$
Dec-Fail	$\dfrac{P \wedge (\underline{n}\ e_1^1 \dots e_{p_1}^1) =^?_{\lambda\sigma} (\underline{m}\ e_1^2 \dots e_{p_2}^2)}{Fail}$, if $m \neq n$.
Exp-λ	$\dfrac{P}{\exists(A \cdot \Gamma \vdash Y : B), P \wedge X =^?_{\lambda\sigma} \lambda_A.Y}$ if $(\Gamma \vdash X : A \to B) \in \mathcal{TV}ar(P)$, $Y \notin \mathcal{TV}ar(P)$, and X is not a solved variable.
Exp-App	$\dfrac{P \wedge X[a_1 \cdot \dots \cdot a_p \cdot \uparrow^n] =^?_{\lambda\sigma} (\underline{m}\ b_1 \dots b_q)}{P \wedge X[a_1 \cdot \dots \cdot a_p \cdot \uparrow^n] =^?_{\lambda\sigma} (\underline{m}\ b_1 \dots b_q) \wedge \bigvee_{r \in R_p \cup R_i} \exists H_1 \dots \exists H_k : X =^?_{\lambda\sigma} (\underline{r}\ H_1 \dots H_k)}$ if X has an atomic type and is not solved; where $H_1, \dots, H_k$ are fresh variables of appropriate types, not occurring in P, with the contexts $\Gamma_{H_i} = \Gamma_X$, R_p is the subset of $\{1, \dots, p\}$ such that $(\underline{r}\ H_1 \dots H_k)$ has the right type, R_i =if $m \geq n+1$ then $\{m - n + p\}$ else $\emptyset$.
Normalise	$\dfrac{P \wedge e_1 =^?_{\lambda\sigma} e_2}{P \wedge e_1' =^?_{\lambda\sigma} e_2'}$ if e_1 or e_2 is not in η-lnf. where e_1' (resp. e_2') is the η-lnf of e_1 (resp. e_2) if e_1 (resp. e_2) is not a solved variable and e_1 (resp. e_2) otherwise.
Replace	$\dfrac{P \wedge X =^?_{\lambda\sigma} t}{\{X \mapsto t\}(P) \wedge X =^?_{\lambda\sigma} t}$ if $X \in \mathcal{TV}ar(P), X \notin \mathcal{TV}ar(t)$ and if t is a meta-variable then $t \in \mathcal{TV}ar(P)$.

Table 2. Unification Rules for the $\lambda\sigma$-calculus

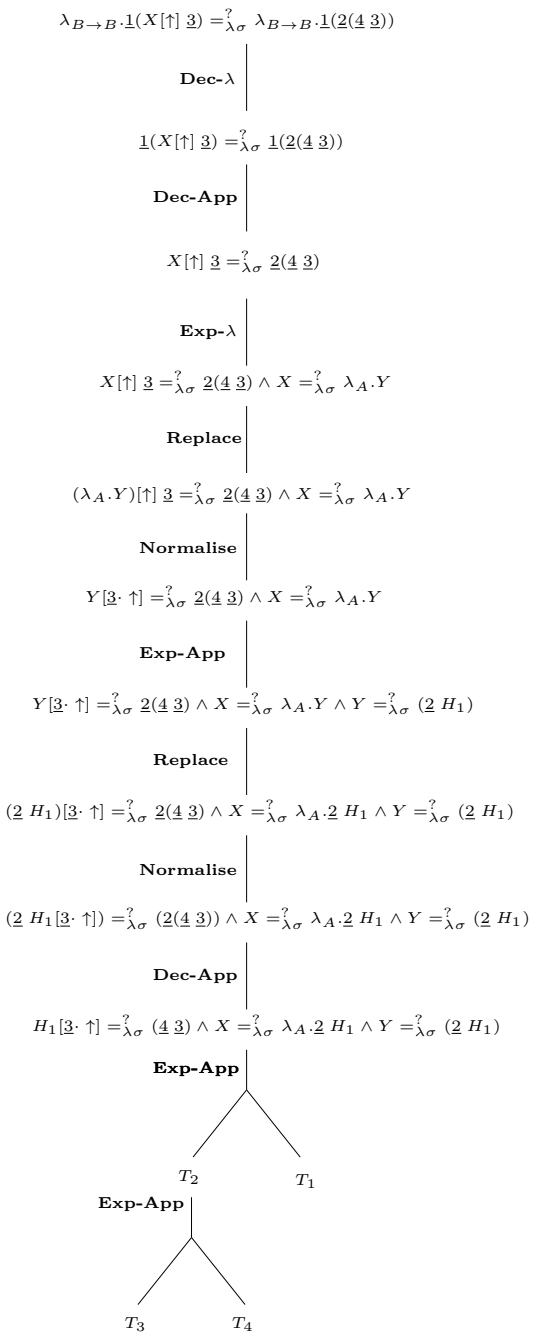


Fig. 5. Example 35: Derivation Tree of $\lambda_{B\to B}.\underline{1}(X\ \underline{3}) =^? \lambda_{B\to B}.\underline{1}(\underline{2}(\underline{4}\ \underline{3}))$

$$H_1[\underline{3}\cdot\uparrow] =^?_{\lambda\sigma} (\underline{4}\ \underline{3}) \wedge X =^?_{\lambda\sigma} \lambda_A.\underline{2}\ H_1 \wedge Y =^?_{\lambda\sigma} (\underline{2}\ H_1) \wedge H_1 =^?_{\lambda\sigma} \underline{1}$$

Replace

$$\underline{1}[\underline{3}\cdot\uparrow] =^?_{\lambda\sigma} (\underline{4}\ \underline{3}) \wedge X =^?_{\lambda\sigma} \lambda_A.\underline{2}\ \underline{1} \wedge Y =^?_{\lambda\sigma} (\underline{2}\ \underline{1}) \wedge H_1 =^?_{\lambda\sigma} \underline{1}$$

Normalise

$$\underline{3} =^?_{\lambda\sigma} (\underline{4}\ \underline{3}) \wedge X =^?_{\lambda\sigma} \lambda_A.\underline{2}\ \underline{1} \wedge Y =^?_{\lambda\sigma} (\underline{2}\ \underline{1}) \wedge H_1 =^?_{\lambda\sigma} \underline{1}$$

Dec-Fail

Fail

Fig. 6. Example 35: subtree T_1 of Fig. 5

$$H_1[\underline{3}\cdot\uparrow] =^?_{\lambda\sigma} (\underline{4}\ \underline{3}) \wedge X =^?_{\lambda\sigma} \lambda_A.\underline{2}\ H_1 \wedge Y =^?_{\lambda\sigma} (\underline{2}\ H_1) \wedge H_1 =^?_{\lambda\sigma} (\underline{4}\ H_2)$$

Replace

$$(\underline{4}\ H_2)[\underline{3}\cdot\uparrow] =^?_{\lambda\sigma} (\underline{4}\ \underline{3}) \wedge X =^?_{\lambda\sigma} \lambda_A.\underline{2}\ (\underline{4}\ H_2) \wedge Y =^?_{\lambda\sigma} (\underline{2}\ (\underline{4}\ H_2)) \wedge H_1 =^?_{\lambda\sigma} (\underline{4}\ H_2)$$

Normalise

$$(\underline{4}\ H_2[\underline{3}\cdot\uparrow]) =^?_{\lambda\sigma} (\underline{4}\ \underline{3}) \wedge X =^?_{\lambda\sigma} \lambda_A.\underline{2}\ (\underline{4}\ H_2) \wedge Y =^?_{\lambda\sigma} (\underline{2}\ (\underline{4}\ H_2)) \wedge H_1 =^?_{\lambda\sigma} (\underline{4}\ H_2)$$

Dec-App

$$H_2[\underline{3}\cdot\uparrow] =^?_{\lambda\sigma} \underline{3} \wedge X =^?_{\lambda\sigma} \lambda_A.\underline{2}\ (\underline{4}\ H_2) \wedge Y =^?_{\lambda\sigma} (\underline{2}\ (\underline{4}\ H_2)) \wedge H_1 =^?_{\lambda\sigma} (\underline{4}\ H_2)$$

Fig. 7. Example 35: subtree T_2 of Fig. 5

$$H_2[\underline{3}\cdot\uparrow] =^?_{\lambda\sigma} \underline{3} \wedge X =^?_{\lambda\sigma} \lambda_A.\underline{2}\ (\underline{4}\ H_2) \wedge Y =^?_{\lambda\sigma} (\underline{2}\ (\underline{4}\ H_2)) \wedge H_1 =^?_{\lambda\sigma} (\underline{4}\ H_2) \wedge H_2 =^?_{\lambda\sigma} \underline{3}$$

Replace

$$\underline{3}[\underline{3}\cdot\uparrow] =^?_{\lambda\sigma} \underline{3} \wedge X =^?_{\lambda\sigma} \lambda_A.\underline{2}\ (\underline{4}\ \underline{3}) \wedge Y =^?_{\lambda\sigma} (\underline{2}\ (\underline{4}\ \underline{3})) \wedge H_1 =^?_{\lambda\sigma} (\underline{4}\ \underline{3}) \wedge H_2 =^?_{\lambda\sigma} \underline{3}$$

Normalise

$$X =^?_{\lambda\sigma} \lambda_A.\underline{2}\ (\underline{4}\ \underline{3}) \wedge Y =^?_{\lambda\sigma} (\underline{2}\ (\underline{4}\ \underline{3})) \wedge H_1 =^?_{\lambda\sigma} (\underline{4}\ \underline{3}) \wedge H_2 =^?_{\lambda\sigma} \underline{3}$$

Success

Fig. 8. The subtree T_4 of Fig. 5.

$$H_2[\underline{3}\cdot\uparrow] =^?_{\lambda\sigma} \underline{3} \wedge X =^?_{\lambda\sigma} \lambda_A.\underline{2}\ (\underline{4}\ H_2) \wedge Y =^?_{\lambda\sigma} (\underline{2}\ (\underline{4}\ H_2)) \wedge H_1 =^?_{\lambda\sigma} (\underline{4}\ H_2) \wedge H_2 =^?_{\lambda\sigma} \underline{1}$$

Replace

$$\underline{1}[\underline{3}\cdot\uparrow] =^?_{\lambda\sigma} \underline{3} \wedge X =^?_{\lambda\sigma} \lambda_A.\underline{2}\ (\underline{4}\ \underline{1}) \wedge Y =^?_{\lambda\sigma} (\underline{2}\ (\underline{4}\ \underline{1})) \wedge H_1 =^?_{\lambda\sigma} (\underline{4}\ \underline{1}) \wedge H_2 =^?_{\lambda\sigma} \underline{1}$$

Normalise

$$X =^?_{\lambda\sigma} \lambda_A.\underline{2}\ (\underline{4}\ \underline{1}) \wedge Y =^?_{\lambda\sigma} (\underline{2}\ (\underline{4}\ \underline{1})) \wedge H_1 =^?_{\lambda\sigma} (\underline{4}\ \underline{1}) \wedge H_2 =^?_{\lambda\sigma} \underline{1}$$

Success

Fig. 9. Example 35: subtree T_3 of Fig. 5

Definition 36 (Pseudo-Precooking). *Let Γ be a context and a be a λ-term well typed in context Γ. To each meta-variable X of type B in the term a, we associate the type B and the context Γ. The pseudo-precooking of the term a from $\Lambda_{dB}(\mathcal{X})$ to $\Lambda_{\lambda\sigma}(\mathcal{X})$ is defined by $\overline{a} = p(a, 0)$, where $p(a, n)$ is defined, for $m \geq 0$, as follows:*

- $p(\lambda_{A_1} \dots \lambda_{A_m}.a, n) = \lambda_{A_1} \dots \lambda_{A_m}.p(a, n + m)$;
- $p(\underline{k}\ a_1 \dots a_m, n) = \underline{1}[\uparrow^{k-1}]\ p(a_1, n) \dots p(a_m, n)$;
- $p(X\ a_1 \dots a_m, n) = Y[p(a_m, n) \cdot \dots \cdot p(a_1, n) \cdot \uparrow^n]$, *where Y is a fresh meta-variable with the target type of X.*

The pseudo-precooking is a function that takes a well typed λ-term a and returns a well typed $\lambda\sigma$-term $\overline{a}$. Intuitively, $\overline{a}$ can be obtained from a_F after normalisation w.r.t. the rules **Exp-λ**, **Replace** and **Normalise** applied to the unification equation that contains a_F.

Example 37. In the unification tree of Fig. 4, take the subproblem $\overline{P}_\epsilon$:

$$\lambda_{B\to B}.X\ \underline{3} =^{?} \lambda_{B\to B}.\underline{2}(\underline{4}\ \underline{3}) \tag{8}$$

whose pseudo-precooking translation is given by $\lambda_{B\to B}.Z[\underline{3} \cdot \uparrow] =^{?}_{\lambda\sigma} \lambda_{B\to B}.\underline{2}(\underline{4}\ \underline{3})$ which can be found in Fig. 5 after the first application of **Normalise** up to the renaming of meta-variables and without the external abstractors. In fact, external abstractors are usually removed by applications of **Dec-λ** at the beginning of the derivation and by analysing each unification rule it is easy to check that no new external abstractors can be introduced during the derivation. This is a consequence of the fact that terms are assumed to be in η-lnf. This pseudo-precooking translation can be obtained from the precooked translation of equation (8) as follows:

$$\dfrac{\dfrac{\dfrac{\lambda_{B\to B}.X[\uparrow]\ \underline{3} =^{?}_{\lambda\sigma} \lambda_{B\to B}.\underline{2}(\underline{4}\ \underline{3})}{\lambda_{B\to B}.X[\uparrow]\ \underline{3} =^{?}_{\lambda\sigma} \lambda_{B\to B}.\underline{2}(\underline{4}\ \underline{3}) \wedge X =^{?}_{\lambda\sigma} \lambda_A.Z}\ \textbf{Exp-}\lambda}{\lambda_{B\to B}.(\lambda_A.Z)[\uparrow]\ \underline{3} =^{?}_{\lambda\sigma} \underline{2}(\underline{4}\ \underline{3}) \wedge X =^{?}_{\lambda\sigma} \lambda_{B\to B}.\lambda_A.Z}\ \textbf{Replace}}{\lambda_{B\to B}.Z[\underline{3} \cdot \uparrow] =^{?}_{\lambda\sigma} \lambda_{B\to B}.\underline{2}(\underline{4}\ \underline{3}) \wedge X =^{?}_{\lambda\sigma} \lambda_A.Z}\ \textbf{Normalise}$$

Applications of **Exp-λ** introduce new equations, but these new equations are ignored by the pseudo-precooking because we are interested only in the structure of the equations obtained in the $\lambda\sigma$-calculus; the information "lost" from these equations is, in some sense, stored in the context of the equation and can be recovered after the application of the strategy **Back** defined in Table 3. An application of **Back** is given in Example 42.

The next proposition shows that the pseudo-precooking translation preserves the types and contexts of the terms.

Proposition 38. *Let Γ be a context and a be a λ-term in $\Lambda_{dB}(\mathcal{X})$ which is well typed in Γ. If a' is a subterm of a such that $A_1 \cdot \dots \cdot A_n \cdot \Gamma \vdash a' : A$ then $A_1 \cdot \dots \cdot A_n \cdot \Gamma \vdash p(a', n) : A$, for $n \geq 0$.*

Proof. The proof is by induction on the structure of a':

- If a' is a de Bruijn index or an application the result is straightforward.
- If $a' = \lambda_B.b$ is a term of type $B \to C$ then by hypothesis we have that $A_1 \cdot \ldots \cdot A_n \cdot \Gamma \vdash \lambda_B.b : B \to C$, and hence $B \cdot A_1 \cdot \ldots \cdot A_n \cdot \Gamma \vdash b : C$. By the induction hypothesis (IH) we have that $B \cdot A_1 \cdot \ldots \cdot A_n \cdot \Gamma \vdash p(b, n+1) : C$. After one application of $(lambda)$ we get that $A_1 \cdot \ldots \cdot A_n \cdot \Gamma \vdash \lambda_B.p(b, n+1) : B \to C$ which is equivalent to $A_1 \cdot \ldots \cdot A_n \cdot \Gamma \vdash p(\lambda_B.b, n) : B \to C$.
- If $a = (X\ b_1 \ldots b_m)$, where X is a meta-variable of type $B_1 \to \ldots \to B_m \to A$ then by hypothesis we have that $A_1 \cdot \ldots \cdot A_n \cdot \Gamma \vdash (X\ a_1 \ldots a_m) : A$. By IH we have that, for all $1 \leq i \leq m$: $A_1 \cdot \ldots \cdot A_n \cdot \Gamma \vdash p(a_i, n) : B_i$. Let Y be a fresh meta-variable of type A and, consider the following derivation:

$$\dfrac{\boxdot \qquad \dfrac{}{B_m \cdot \ldots \cdot B_1 \cdot \Gamma \vdash Y : A}\,(meta)}{A_1 \cdot \ldots \cdot A_n \cdot \Gamma \vdash Y[p(b_m, n) \cdot \ldots \cdot p(b_1, n) \cdot \uparrow^n] : A}\,(clos)$$

where $\boxdot$ corresponds to

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A_1 \cdot \ldots \cdot A_n \cdot \Gamma \vdash p(b_1, n) : B_1}\,(IH) \quad \dfrac{}{A_1 \cdot \ldots \cdot A_n \cdot \Gamma \vdash \uparrow^n \rhd \Gamma}\,(shift)}{A_1 \cdot \ldots \cdot A_n \cdot \Gamma \vdash p(b_1, n) \cdot \uparrow^n \rhd B_1 \cdot \Gamma}\,(cons)}{}\,(cons)}{\vdots \qquad\qquad \vdots}\,(clos)$$
$$A_1 \cdot \ldots \cdot A_n \cdot \Gamma \vdash p(b_m, n) \cdot \ldots \cdot p(b_1, n) \cdot \uparrow^n \rhd B_m \cdot \ldots \cdot B_1 \cdot \Gamma$$

□

(* add motivation *)

Lemma 39. *Let P be a unification problem in the simply typed λ-calculus. The pseudo-precooking of the terms of P is obtained from P_F by normalising these terms w.r.t. the rules* **Exp-λ**, **Replace** *and* **Normalise**.

Proof. The proof is by induction on the size of the type of the meta-variables occurring in P. Without loss of generality, we prove this Lemma for the case where P contains occurrences of only one meta-variable, say X. Unification problems that contains occurrences of different meta-variables are proven in the same way because applications of **Exp-λ** consider one meta-variable at a time and replacement of one meta-variable do not affected other meta-variables with different names.

Let

$$(X[\uparrow^n]\ a_1 \ldots a_p) \tag{9}$$

be a sub-term of P_F, where the terms $a_1, \ldots, a_p$ may contain occurrences of X. If X is atomic, i.e., $p = 0$ then the result follows vacuously. If $p = 1$ then the term (9) assumes the form $(X[\uparrow^n]\ a_1)$, for some $n \geq 0$. If a_1 contains occurrences of X then all these occurrences have also have the form $(X[\uparrow^m]\ b_1)$, for some $m \geq 0$. Moreover, if X has type $B \to A$ then a_1 and b_1 are terms of type B.

The next lemma shows formally how unification problems in the λ-calculus are related to unification systems in the $\lambda\sigma$-calculus.

Lemma 40. *Let Γ be a context, P be a unification problem in $\Lambda_{dB}(\mathcal{X})$ well typed in Γ and $\mathcal{A}(P)$ its unification tree. For each subproblem P_α occurring in $\mathcal{A}(P)$, there exists a unification system P^* derived from P_F via* **Unif** *such that:*

1. *For each equation in P_α of the form*

$$\lambda_{A_1} \dots \lambda_{A_n}.h_1\ e_1^1 \dots e_{p_1}^1 =^? \lambda_{A_1} \dots \lambda_{A_n}.\underline{h}\ e_1^2 \dots e_{p_2}^2 \tag{10}$$

well typed in context Γ, where h_1 is either a de Bruijn index or a meta-variable, there exists, respectively, an equation in P^* either of the form:

$$h_1\ \overline{e}_1^1 \dots \overline{e}_{p_1}^1 =^?_{\lambda\sigma} \underline{h}\ \overline{e}_1^2 \dots \overline{e}_{p_2}^2 \qquad \text{if } h_1 \text{ is a de Bruijn index,} \tag{11}$$

or

$$Y[\overline{e}_{p_1}^1 \cdot \dots \cdot \overline{e}_1^1 \cdot \uparrow^n] =^?_{\lambda\sigma} \underline{h}\ \overline{e}_1^2 \dots \overline{e}_{p_2}^2 \qquad \text{if } h_1 \text{ is a meta-variable;} \tag{12}$$

that is well typed in context $A_1 \cdot \dots \cdot A_n \cdot \Gamma$.

2. *For each flexible-flexible equation eq in P_α, the equation eq_F is in P^*. In this case, we assume that no unification rule is applied to flexible-flexible equations since we do not need to deal with them.*

Proof. The proof is by induction on the length of the derivation that generates P_α. If $\alpha = \epsilon$ then for each equation eq in P, eq_F is in P_F and, if P_ϵ contains only flexible-flexible equations, we take $P^* = P_F$. If P_ϵ contains flexible-rigid or rigid-rigid equations, i.e., equations of the form:

$$\lambda_{A_1} \dots \lambda_{A_n}.h_1\ e_1^1 \dots e_{p_1}^1 =^? \lambda_{A_1} \dots \lambda_{A_n}.\underline{h}\ e_1^2 \dots e_{p_2}^2$$

where h_1 is either a meta-variable or a de Bruijn index, then we consider each case separately:

• If h_1 is a meta-variable, say X, then P_F contains the equation

$$\lambda_{A_1} \dots \lambda_{A_n}.X[\uparrow^n]\ e_{1_F}^1 \dots e_{p_{1_F}}^1 =^?_{\lambda\sigma} \lambda_{A_1} \dots \lambda_{A_n}.\underline{h}\ e_{1_F}^2 \dots e_{p_{2_F}}^2$$

and we consider the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\lambda_{A_1} \dots \lambda_{A_n}.X[\uparrow^n]\ e_{1_F}^1 \dots e_{p_{1_F}}^1 =^?_{\lambda\sigma} \lambda_{A_1} \dots \lambda_{A_n}.\underline{h}\ e_{1_F}^2 \dots e_{p_{2_F}}^2}{\vdots\ n \text{ times}}\ \textbf{Dec-}\lambda}{X[\uparrow^n]\ e_{1_F}^1 \dots e_{p_{1_F}}^1 =^?_{\lambda\sigma} \underline{h}\ e_{1_F}^2 \dots e_{p_{2_F}}^2}\ \textbf{Dec-}\lambda}{X[\uparrow^n]\ e_{1_F}^1 \dots e_{p_{1_F}}^1 =^?_{\lambda\sigma} \underline{h}\ e_{1_F}^2 \dots e_{p_{2_F}}^2 \wedge X =^?_{\lambda\sigma} \lambda_{B_1}.X_1}\ \textbf{Exp-}\lambda}{(\lambda_{B_1}.X_1)[\uparrow^n]\ e'^1_1 \dots e'^1_{p_1} =^?_{\lambda\sigma} \underline{h}\ e'^2_1 \dots e'^2_{p_2} \wedge X =^?_{\lambda\sigma} \lambda_{B_1}.X_1}\ \textbf{Replace}}{\vdots\ (p_1 - 1) \text{ times}}\ \boxdot$$

$$\dfrac{(\lambda_{B_1} \dots \lambda_{B_{p_1}}.Y)[\uparrow^n]\ \tilde{e}_1^1 \dots \tilde{e}_{p_1}^1 =^?_{\lambda\sigma} \underline{h}\ \tilde{e}_1^2 \dots \tilde{e}_{p_2}^2 \wedge X =^?_{\lambda\sigma} \lambda_{B_1} \dots \lambda_{B_{p_1}}.Y \wedge \dots}{Y[\overline{e}_{p_1}^1 \cdot \dots \cdot \overline{e}_1^1 \cdot \uparrow^n] =^?_{\lambda\sigma} \underline{h}\ \overline{e}_1^2 \dots \overline{e}_{p_2}^2 \wedge X =^?_{\lambda\sigma} \lambda_{B_1} \dots \lambda_{B_{p_1}}.Y \wedge \dots}\ \boxtimes$$

where the terms ${e'}^1_1, \ldots, {e'}^1_{p_1}, {e'}^2_1, \ldots, {e'}^2_{p_2}$ are obtained respectively from $e^1_{1_F}, \ldots, e^1_{p_{1_F}}, e^2_{1_F}, \ldots, e^2_{p_{2_F}}$ by replacing each occurrence of X for $\lambda_{B_1}.X_1$, $\boxdot$ corresponds to an application of **Exp-λ** followed by an application of **Replace**, Y is a fresh meta-variable, $\boxtimes$ corresponds to an application of **Normalise** and the terms . After the above derivation, we apply **Exp-λ** followed by **Replace** to every sub-term of $e^1_{1_F}, \ldots, e^1_{p_{1_F}}, e^2_{1_F}, \ldots, e^2_{p_{2_F}}$ of the form $(V[\uparrow^m]\ f_1 \ldots f_r)$ $(m, r \geq 0)$ recursively. After a normalisation step, we get sub-terms of the form $W[\overline{f}_r \cdot \ldots \cdot \overline{f}_1 \cdot \uparrow^m]$, where W is a fresh meta-variable with the target type of V. In this way, we get a unification system containing the equation $Y[\overline{e}^1_{p_1} \cdot \ldots \cdot \overline{e}^1_1 \cdot \uparrow^n] =^?_{\lambda\sigma} \underline{h}\ \overline{e}^2_1 \ldots \overline{e}^2_{p_2}$. All the other occurrences of X in the current unification system are replaced by $\lambda_{B_1} \ldots \lambda_{B_{p_1}}.Y$ and, hence all the terms that have head X remain flexible and, after a replacement X becomes a solved meta-variable. During the above derivation, no trivial equation can be generated because applications of **Exp-λ** always introduce fresh meta-variables that cannot be eliminated by further applications of **Dec-λ**, **Replace** or **Normalise** that correspond to the rules used in the above derivation.

The above derivation shows that after n applications of **Dec-λ** and a finite number of applications of **Exp-λ** always followed by an application of **Replace** we get a problem (after a normalization step) containing an equation with the same structure of equation (12). The order the rules **Dec-λ** and **Exp-λ** are applied in the above derivation is irrelevant because they are applied in different positions of the term. The only restriction is that **Exp-λ** is always followed by an application of **Replace** which avoids infinite applications of **Exp-λ** (cf. [DHK00]). In addition, we assume at least one application of **Normalise** at the end of the derivation, but intermediate normalization steps can be applied and do not change the resulting term.

• If h_1 is a de Bruijn index then P_F contains the equation

$$\lambda_{A_1} \ldots \lambda_{A_n}.h_1\ e^1_{1_F} \ldots e^1_{p_{1_F}} =^?_{\lambda\sigma} \lambda_{A_1} \ldots \lambda_{A_n}.\underline{h}\ e^2_{1_F} \ldots e^2_{p_{2_F}}.$$

We take P^* to be the problem obtained from P_F after n applications of **Dec-λ** followed by applications **Exp-λ** and **Replace** to every sub-term of $e^1_{1_F}, \ldots, e^1_{p_{1_F}}, e^2_{1_F}, \ldots, e^2_{p_{2_F}}$ of the form $(V[\uparrow^m]\ f_1 \ldots f_r)$. As in the previous case, no equation is eliminated during this process.

For the induction step, let P_α be a unification problem in $\mathcal{A}(P)$ with $\alpha \neq \epsilon$ and let P^* be the unification system obtained from P_F given by this lemma; we need to find a unification system, say P^{**}, derived from P^* that satisfies this lemma for the problem derived from P_α in one step. We consider 2 cases:

• *The unification problem derived from P_α is obtained after an application of SIMPL:*

In this case, the unification problem derived from P_α is $\overline{P}_\alpha$ according to the definition of unification trees and, the analysis over $\overline{P}_\alpha$ is divided in two cases:

1. $\overline{P}_\alpha$ contains flexible-rigid or rigid-rigid equations: Then P_α contains (at least) one rigid-rigid equation of the form

$$\lambda_{B_1} \dots \lambda_{B_m}.\underline{k}\ f_1^1 \dots f_p^1 =^? \lambda_{B_1} \dots \lambda_{B_m}.\underline{k}\ f_1^2 \dots f_p^2 \tag{13}$$

well typed in context Γ and such that, for some $i = 1, \dots, p$:

$$f_i^1 = \lambda_{C_1} \dots \lambda_{C_w}.h_1\ e_1^1 \dots e_{p_1}^1 \text{ and } f_i^2 = \lambda_{C_1} \dots \lambda_{C_w}.\underline{h}\ e_1^2 \dots e_{p_2}^2$$

where h_1 is a de Bruijn index or a meta-variable.

An application of SIMPL to the current problem replaces equation (13) by the conjunction of equations:

$\lambda_{B_1} \dots \lambda_{B_m}.f_1^1 =^? \lambda_{B_1} \dots \lambda_{B_m}.f_1^2 \wedge \dots \wedge \lambda_{B_1} \dots \lambda_{B_m}.f_i^1 =^? \lambda_{B_1} \dots \lambda_{B_m}.f_i^2 \wedge \dots \wedge \lambda_{B_1} \dots \lambda_{B_m}.f_p^1 =^? \lambda_{B_1} \dots \lambda_{B_m}.f_p^2$ that are well typed in context Γ and, the equation $\lambda_{B_1} \dots \lambda_{B_m}.f_i^1 =^? \lambda_{B_1} \dots \lambda_{B_m}.f_i^2$ has the form:

$$\lambda_{A_1} \dots \lambda_{A_n}.h_1\ e_1^1 \dots e_{p_1}^1 =^? \lambda_{A_1} \dots \lambda_{A_n}.\underline{h}\ e_1^2 \dots e_{p_2}^2$$

where $A_i = \begin{cases} B_i & \text{, for } 1 \leq i \leq m; \\ C_{i-m} & \text{, for } m < i \leq n (= m + w) \end{cases}$
and h_1 is either a de Bruijn index or a meta-variable.

By hypothesis, there exists a derivation from P_F that generates a unification system P^* containing a rigid-rigid equation of the form:

$$\underline{k}\ \overline{f}_1^1 \dots \overline{f}_i^1 \dots \overline{f}_p^1 =^?_{\lambda\sigma} \underline{k}\ \overline{f}_1^2 \dots \overline{f}_i^2 \dots \overline{f}_p^2$$

which is well typed in context $B_1 \cdot \ldots \cdot B_m \cdot \Gamma$. After an application of **Dec-App**, the previous rigid-rigid equation is replaced by the conjunction $\overline{f}_1^1 =^?_{\lambda\sigma} \overline{f}_1^2 \wedge \dots \wedge \overline{f}_i^1 =^?_{\lambda\sigma} \overline{f}_i^2 \wedge \dots \wedge \overline{f}_p^1 =^?_{\lambda\sigma} \overline{f}_p^2$. We consider 2 sub-cases:

(a) h_1 is a de Bruijn index: In this case, the equation $\overline{f}_i^1 =^?_{\lambda\sigma} \overline{f}_i^2$ has the form

$$\lambda_{C_1} \dots \lambda_{C_w}.h_1\ \overline{e}_1^1 \dots \overline{e}_{p_1}^1 =^?_{\lambda\sigma} \lambda_{C_1} \dots \lambda_{C_w}.\underline{h}\ \overline{e}_1^2 \dots \overline{e}_{p_2}^2.$$

After w applications of **Dec-λ**, we get a system with the desired equation, i.e., with the equation

$$h_1\ \overline{e}_1^1 \dots \overline{e}_{p_1}^1 =^?_{\lambda\sigma} \underline{h}\ \overline{e}_1^2 \dots \overline{e}_{p_2}^2$$

well typed in context $B_1 \cdot \ldots \cdot B_m \cdot C_1 \cdot \ldots \cdot C_w \cdot \Gamma$. Notice that during this process all other equations remains unchanged because there is no substitution involved in these steps.

(b) h_1 is a meta-variable: In this case, the equation $\overline{f}_i^1 =^?_{\lambda\sigma} \overline{f}_i^2$ has the form $\lambda_{C_1} \dots \lambda_{C_w}.Y[\overline{e}_{p_1}^1 \cdot \ldots \cdot \overline{e}_1^1 \cdot \uparrow^{m+w}] =^?_{\lambda\sigma} \lambda_{C_1} \dots \lambda_{C_w}.\underline{h}\ \overline{e}_1^2 \dots \overline{e}_{p_2}^2$, and after w applications of **Dec-λ** to this equation we get the desired equation well typed in context $B_1 \cdot \ldots \cdot B_m \cdot C_1 \cdot \ldots \cdot C_w \cdot \Gamma$.

Take P^{**} to be the unification system obtained from P^* after all possible applications of **Dec-App** followed by the strategy described in items *(a)* and

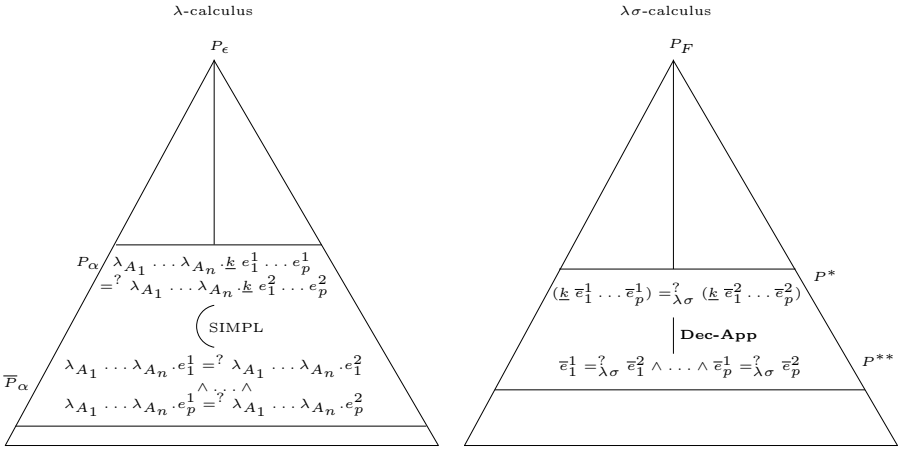


Fig. 10. A Simplification Step

(b) above.

2. $\overline{P}_\alpha$ contains only flexible-flexible equations: Then P_α contains (at least) one equation of the form

$$\lambda_{A_1} \ldots \lambda_{A_n} . \underline{k}\ e_1^1 \ldots e_p^1 =^? \lambda_{A_1} \ldots \lambda_{A_n} . \underline{k}\ e_1^2 \ldots e_p^2$$

well typed in context Γ, and where all the terms $e_1^1, \ldots, e_p^1, \ldots, e_1^2, \ldots, e_p^2$ are flexible. The application of SIMPL to P_α replaces the previous rigid-rigid equation by the conjunction $\lambda_{A_1} \ldots \lambda_{A_n} . e_1^1 =^? \lambda_{A_1} \ldots \lambda_{A_n} . e_1^2 \wedge \ldots \wedge \lambda_{A_1} \ldots \lambda_{A_n} . e_p^1 =^? \lambda_{A_1} \ldots \lambda_{A_n} . e_p^2$ where every equation is well typed in context Γ and, leaves all the other equations unchanged.

By hypothesis, there exists a derivation of P_F that generates the problem P^* that contains the equation $\underline{k}\ \overline{e}_1^1 \ldots \overline{e}_p^1 =^?_{\lambda\sigma} \underline{k}\ \overline{e}_1^2 \ldots \overline{e}_p^2$ well typed in context $A_1 \cdot \ldots \cdot A_n \cdot \Gamma$ and, where all terms $\overline{e}_1^1, \ldots, \overline{e}_p^1, \overline{e}_1^2, \ldots, \overline{e}_p^2$ are flexible because the precooking translation carry flexible terms into flexible terms and, applications of **Exp-λ** and **Replace** just replace meta-variables by fresh meta-variables that are under the scope of new abstractors. Therefore, after an application of **Dec-App** we get the conjunction $\overline{e}_1^1 =^?_{\lambda\sigma} \overline{e}_1^2 \wedge \ldots \wedge \overline{e}_p^1 =^?_{\lambda\sigma} \overline{e}_p^2$ where all the equations are well typed in context $A_1 \cdot \ldots \cdot A_n \cdot \Gamma$ and all the other equations in the system remain unchanged. This process is repeated for all rigid-rigid equations in P_α and, consequently in P^*. Notice that equations that are not rigid-rigid remain unchanged. Take the current system to be P^{**} (see Fig. 10).

• *The unification problem derived from P_α is obtained after an application of MATCH:*

Let $P_{\alpha r}$ $(r > 0)$ be the unification problem generated after this application of MATCH. The problem $P_{\alpha r}$ must contain at least one equation that is either

flexible-rigid or rigid-rigid (that may be trivial) because after an imitation step the resulting equation is rigid-rigid and, after a projection, it is either flexible-rigid or rigid-rigid.

Assume that P_α contains (at least) one flexible-rigid equation of the form

$$\lambda_{A_1} \dots \lambda_{A_n}.X\ e_1^1 \dots e_{p_1}^1 =^? \lambda_{A_1} \dots \lambda_{A_n}.\underline{h}\ e_1^2 \dots e_{p_2}^2 \tag{14}$$

well typed in context Γ and where:

- $n, p_1, p_2 \geq 0$;
- X is a meta-variable with type $B_1 \to \ldots \to B_{p_1} \to A$ with A atomic;
- $\underline{h}$ is a de Bruijn index with type $C_1 \to \ldots \to C_{p_2} \to A$ with A atomic;
- If $p_1 > 0$ then e_i^1 are λ-terms in η-lnf with type B_i for all $1 \leq i \leq p_1$.
- If $p_2 > 0$ then e_j^2 are λ-terms in η-lnf with type C_j, for all $1 \leq j \leq p_2$.

We consider the imitation and projection substitutions separately:

(a) Imitation: An imitation is possible only if the head of the rigid term is a constant, i.e., when $h > n$. In this case, the imitation substitution is given by:

$$X/\lambda_{B_1} \dots \lambda_{B_{p_1}}.\underline{h-n+p_1}\ (H_1\ \underline{p_1} \dots \underline{1}) \dots (H_{p_2}\ \underline{p_1} \dots \underline{1})$$

where the H_i's are fresh meta-variables with types $B_1 \to \ldots \to B_{p_1} \to C_i$ for all $1 \leq i \leq p_2$. After an application of this substitution to equation (14) we get the unification problem $P_{\alpha r}$ that contains the following equation:

$$\lambda_{A_1} \dots \lambda_{A_n}.\underline{h}\ (H_1\ e_1^1 \dots e_{p_1}^1) \dots (H_{p_2}\ e_1^1 \dots e_{p_1}^1) =^? \lambda_{A_1} \dots \lambda_{A_n}.\underline{h}\ e_1^2 \dots e_{p_2}^2$$

well typed in context Γ.

By hypothesis, there exists a derivation from P_F that generates the system P^* that contains the equation:

$$Y[\overline{e}_{p_1}^1 \cdot \ldots \cdot \overline{e}_1^1 \cdot \uparrow^n] =^?_{\lambda\sigma} \underline{h}\ \overline{e}_1^2 \dots \overline{e}_{p_2}^2$$

which is well typed in context $A_1 \cdot \ldots \cdot A_n \cdot \Gamma$. Since $h > n$, after an application of **Exp-App**, we get a unification system containing the conjunction:

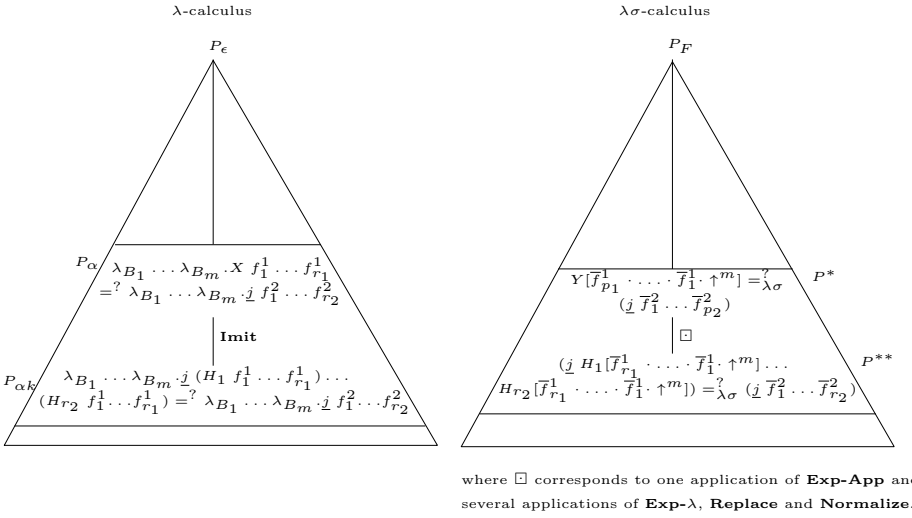
$$Y[\overline{e}_{p_1}^1 \cdot \ldots \cdot \overline{e}_1^1 \cdot \uparrow^n] =^?_{\lambda\sigma} \underline{h}\ \overline{e}_1^2 \dots \overline{e}_{p_2}^2 \wedge Y =^?_{\lambda\sigma} \underline{h-n+p_1}\ W_1 \dots W_{p_2}$$

where the W_j's are fresh meta-variables with type C_j, for all $1 \leq j \leq p_2$.

After an application of **Replace** and then **Normalise** to the current system, we get a new system containing the equation:

$$\underline{h}\ W_1[\overline{e}_{p_1}^1 \cdot \ldots \cdot \overline{e}_1^1 \cdot \uparrow^n] \dots W_{p_2}[\overline{e}_{p_1}^1 \cdot \ldots \cdot \overline{e}_1^1 \cdot \uparrow^n] =^?_{\lambda\sigma} \underline{h}\ \overline{e}_1^2 \dots \overline{e}_{p_2}^2$$

which is well typed in the context $A_1 \cdot \ldots \cdot A_n \cdot \Gamma$. Take P^{**} to be the current system. The general scheme is shown in Fig. 11. There exists only one case when an equation is eliminated during this process: if the de Bruijn index $\underline{h}$ has an atomic type (because in this case no meta-variable is introduced and a trivial



where ⊡ corresponds to one application of **Exp-App** and several applications of **Exp-λ**, **Replace** and **Normalize**.

Fig. 11. The imitation step

equation is generated). But if this is the case, then a trivial equation is also generated in $P_{\alpha r}$ by the same reason and, the lemma holds.

(b) Projection: In this case, the head X of the flexible term is projected over each of its arguments whose target typed is equal to the target type of X. Suppose, without loss of generality, that X is projected over its l-th argument ($1 \leq l \leq p_1$), i.e., suppose that e_l^1 has type of the form $D_1 \to \ldots \to D_q \to A$ ($q \geq 0$). The projection of X over its l-th argument is given by:

$$X/\lambda_{B_1} \dots \lambda_{B_{p_1}}.\underline{p_1 - l + 1}\ (H_1\ \underline{p_1} \dots \underline{1}) \dots (H_q\ \underline{p_1} \dots \underline{1})$$

where the H_i's are fresh meta-variables of type $B_1 \to \ldots \to B_{p_1} \to D_i$ for all $1 \leq i \leq q$. After an application of this substitution, we get a problem that contains the following equation:

$$\lambda_{A_1} \dots \lambda_{A_n}.e_l^1\ (H_1\ e_1^1 \dots e_{p_1}^1) \dots (H_q\ e_1^1 \dots e_{p_1}^1) =^?\ \lambda_{A_1} \dots \lambda_{A_n}.\underline{h}\ e_1^2 \dots e_{p_2}^2 \quad (15)$$

well typed in context Γ. We consider 2 sub-cases:

(b.1) The head of the term e_l^1 is a de Bruijn index: Since e_l^1 is in η-lnf, we may assume without loss of generality, that e_l^1 is of the form $\lambda_{D_1} \dots \lambda_{D_q}.\underline{k}\ f_1 \dots f_s$. After a normalization step we get the unification problem $P_{\alpha r}$ that contains one of the following equations according to the value of k:

- $k > q$:

$$\lambda_{A_1} \dots \lambda_{A_n}.\underline{k - q}\ f_1^1 \dots f_s^1 =^?\ \lambda_{A_1} \dots \lambda_{A_n}.\underline{h}\ e_1^2 \dots e_{p_2}^2$$

• $k \leq q$:

$$\lambda_{A_1} \dots \lambda_{A_n}.H_{q-k+1}\ e_1^1 \dots e_{p_1}^1\ f_1^1 \dots f_s^1 =^? \lambda_{A_1} \dots \lambda_{A_n}.\underline{h}\ e_1^2 \dots e_{p_2}^2$$

well typed in context Γ and where, for all $1 \leq i \leq s$, f_i^1 is obtained from f_i after replacing all free occurrences of $\underline{1}, \dots, \underline{q}$, respectively for $(H_q\ e_1^1 \dots e_{p_1}^1)$, $\dots$, $(H_1\ e_1^1 \dots e_{p_1}^1)$.

By hypothesis, there exists a derivation of P_F that generates a unification system P^* containing the equation

$$Y[\overline{e}_{p_1}^1 \cdot \dots \cdot \overline{e}_1^1 \cdot \uparrow^n] =^?_{\lambda\sigma} \underline{h}\ \overline{e}_1^2 \dots \overline{e}_{p_2}^2 \tag{16}$$

well typed in context $A_1 \cdot \dots \cdot A_n \cdot \Gamma$. The precooking translation preserves types and, hence the target type of $\overline{e}_l^1$ coincides with the type of Y and, an application of **Exp-App** generates a unification system containing the conjunction:

$$Y[\overline{e}_{p_1}^1 \cdot \dots \cdot \overline{e}_1^1 \cdot \uparrow^n] =^?_{\lambda\sigma} \underline{h}\ \overline{e}_1^2 \dots \overline{e}_{p_2}^2 \wedge Y =^?_{\lambda\sigma} (\underline{p_1 - l + 1}\ W_1 \dots W_q)$$

where the W_i's are fresh meta-variables of type D_i for all $1 \leq i \leq q$. After an application of **Replace** and **Normalise**, we get a system containing the following equation:

$$\overline{e}_l^1\ W_1[\overline{e}_{p_1}^1 \cdot \dots \cdot \overline{e}_1^1 \cdot \uparrow^n] \dots W_q[\overline{e}_{p_1}^1 \cdot \dots \cdot \overline{e}_1^1 \cdot \uparrow^n] =^?_{\lambda\sigma} \underline{h}\ \overline{e}_1^2 \dots \overline{e}_{p_2}^2 \tag{17}$$

well typed in context $A_1 \cdot \dots \cdot A_n \cdot \Gamma$. Since $\overline{e}_l^1 = \lambda_{D_1} \dots \lambda_{D_q}.\underline{k}\ \overline{f}_1 \dots \overline{f}_s$, the left hand side of equation (17) reduces as follows according to the value of k:

• $k > q$:

$$(\lambda_{D_1} \dots \lambda_{D_q}.\underline{k}\ \overline{f}_1 \dots \overline{f}_s)\ W_1[\overline{e}_{p_1}^1 \cdot \dots \cdot \overline{e}_1^1 \cdot \uparrow^n] \dots W_q[\overline{e}_{p_1}^1 \cdot \dots \cdot \overline{e}_1^1 \cdot \uparrow^n] \rightarrow^*_{\lambda\sigma}$$

$$\underline{k-q}\ \overline{f}_1[W_q[\overline{e}_{p_1}^1 \cdot \dots \cdot \overline{e}_1^1 \cdot \uparrow^n] \cdot \dots \cdot W_1[\overline{e}_{p_1}^1 \cdot \dots \cdot \overline{e}_1^1 \cdot \uparrow^n] \cdot id] \dots$$
$$\overline{f}_s[W_q[\overline{e}_{p_1}^1 \cdot \dots \cdot \overline{e}_1^1 \cdot \uparrow^n] \cdot \dots \cdot W_1[\overline{e}_{p_1}^1 \cdot \dots \cdot \overline{e}_1^1 \cdot \uparrow^n] \cdot id].$$

• $k \leq q$:

$$(\lambda_{D_1} \dots \lambda_{D_q}.\underline{k}\ \overline{f}_1 \dots \overline{f}_s)\ W_1[\overline{e}_{p_1}^1 \cdot \dots \cdot \overline{e}_1^1 \cdot \uparrow^n] \dots W_q[\overline{e}_{p_1}^1 \cdot \dots \cdot \overline{e}_1^1 \cdot \uparrow^n] \rightarrow^*_{\lambda\sigma}$$

$$W_{q-k+1}[\overline{e}_{p_1}^1 \cdot \dots \cdot \overline{e}_1^1 \cdot \uparrow^n]\ \overline{f}_1[W_q[\overline{e}_{p_1}^1 \cdot \dots \cdot \overline{e}_1^1 \cdot \uparrow^n] \cdot \dots \cdot W_1[\overline{e}_{p_1}^1 \cdot \dots \cdot \overline{e}_1^1 \cdot \uparrow^n] \cdot id] \dots$$
$$\overline{f}_s[W_q[\overline{e}_{p_1}^1 \cdot \dots \cdot \overline{e}_1^1 \cdot \uparrow^n] \cdot \dots \cdot W_1[\overline{e}_{p_1}^1 \cdot \dots \cdot \overline{e}_1^1 \cdot \uparrow^n] \cdot id].$$

Notice that for all $1 \leq j \leq q$, $W_j[\overline{e}_{p_1}^1 \cdot \dots \cdot \overline{e}_1^1 \cdot \uparrow^n]$ corresponds to the pseudo-precooking of the sub-term $(H_j\ e_1^1 \dots e_{p_1}^1)$ of the left hand side of equation (15) that is under the scope of n abstractions. Therefore, for all $1 \leq i \leq s$, the term

$$\overline{f}_i[W_q[\overline{e}_{p_1}^1 \cdot \dots \cdot \overline{e}_1^1 \cdot \uparrow^n] \cdot \dots \cdot W_1[\overline{e}_{p_1}^1 \cdot \dots \cdot \overline{e}_1^1 \cdot \uparrow^n] \cdot id]$$

corresponds to the pseudo-precooking of f_i^1, written $\overline{f}_i^1$, and in this way the unification system obtained so far contains one of the following equations according

to the value of k:

- $k > q$

$$\underline{k-q}\ \overline{f}^1_1 \dots \overline{f}^1_r =^?_{\lambda\sigma} \underline{h}\ \overline{e}^2_1 \dots \overline{e}^2_{p_2}$$

- $k \leq q$

$$M[\overline{f}^1_r \cdot \ldots \cdot \overline{f}^1_1 \cdot \overline{e}^1_{p_1} \cdot \ldots \cdot \overline{e}^1_1 \cdot \uparrow^n] =^?_{\lambda\sigma} \underline{h}\ \overline{e}^2_1 \dots \overline{e}^2_{p_2} \text{ after}$$

applications of **Exp-λ**, **Replace** and **Normalise**;

where M is a meta-variable of type A

which are well typed in context $A_1 \cdot \ldots \cdot A_n \cdot \Gamma$. Take P^{**} to be the current unification system.

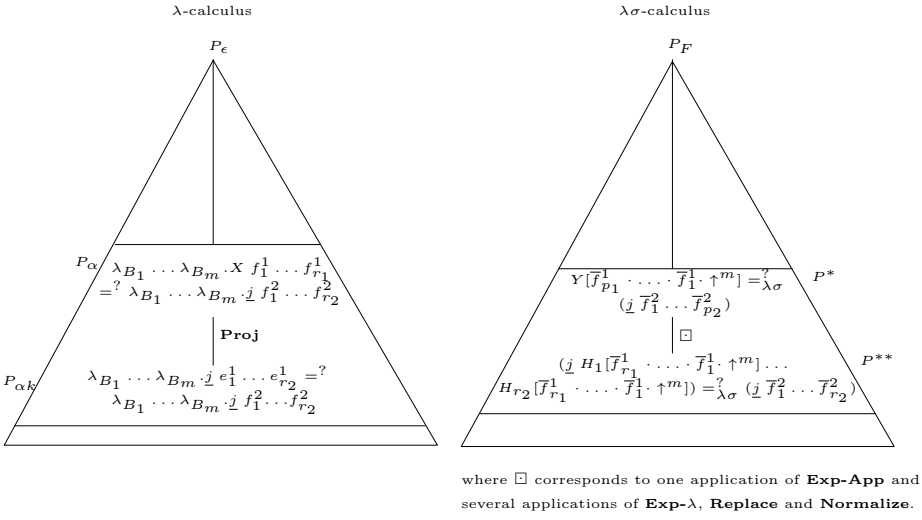


Fig. 12. The projection step

(b.2) The head of the term e^1_l is a meta-variable: In this case e^1_l is of the form $\lambda_{D_1} \dots \lambda_{D_q}.Z\ f_1 \dots f_s$ and normalizing equation (15) we get the unification problem $P_{\alpha k}$ that contains the equation:

$$\lambda_{A_1} \dots \lambda_{A_n}.Z\ f^1_1 \dots f^1_s =^? \lambda_{A_1} \dots \lambda_{A_n}.\underline{h}\ e^2_1 \dots e^2_{p_2}$$

well typed in context Γ and, as in the previous case f^1_j is obtained from f_j by replacing all occurrences of $\underline{1}, \dots, \underline{q}$, respectively by the terms $(H_q\ e^1_1 \dots e^1_{p_1}), \dots, (H_1\ e^1_1 \dots e^1_{p_1})$. By hypothesis, there exists a unification system P^*, derived from P_F and containing equation (16) and after application of **Exp-App**, **Replace**

and **Normalise** we get a new system containing equation (17). Since $\overline{e}_l^1 = \lambda_{D_1} \ldots \lambda_{D_q}.Z[\uparrow^{n+q}]\ \overline{f}_1 \ldots \overline{f}_s$, we conclude that the normal form of equation (17) is given by:

$$U[\overline{f}_s^1 \cdot \ldots \cdot \overline{f}_1^1 \cdot \uparrow^n] =^?_{\lambda\sigma} \underline{h}\ \overline{e}_1^2 \ldots \overline{e}_{p_2}^2 \tag{18}$$

which is well typed in context $A_1 \cdot \ldots \cdot A_n \cdot \Gamma$ and, where U is a fresh meta-variable with the type of the sub-term $(Z[\uparrow^{n+q}]\ \overline{f}_1 \ldots \overline{f}_s)$. Take P^{**} to be the system containing equation (18). □

In Lemma 40, we established a relation between the structure of the equations of the subproblems generated during the unification process in the simply typed λ-calculus in de Bruijn's notation and the precooked translation of these problems in the simply typed $\lambda\sigma$-calculus. This lemma is the key point for the following results that relate the solutions and the subtrees generated during the unification process. In fact, the next proposition shows that if $\mathcal{A}(P)$ is the unification tree of a given unification problem P, then for each sub-tree of $\mathcal{A}(P)$ there exists a sub-tree of the derivation tree of P_F with the same number of success and fail nodes. Moreover, the precooked version of subproblems of P can be obtained as subproblems of P_F. Before giving the proposition, we present another set of rules that in a certain way undo the work done by the rules **Exp-λ** and **Dec-λ**. This set is called **Back** and is given in Table 3 (cf. [DHK00]).

Anti-Exp-λ	$\dfrac{P}{\exists Y(P \wedge X =^?_{\lambda\sigma} (Y[\uparrow]\ \underline{1}))}$ if $X \in Var(P)$ such that $\Gamma_X = A \cdot \Gamma'_X$ where $Y \in \mathcal{X}, Y \notin Var(P)$ and $T_y = A \to T_X, \Gamma_Y = \Gamma'_X$
Anti-Dec-λ	$\dfrac{P \wedge a =^?_{\lambda\sigma} b}{P \wedge \lambda_A.a =^?_{\lambda\sigma} \lambda_A.b}$ if $a =^?_{\lambda\sigma} b$ is well typed in context $\Delta = A \cdot \Delta'$.

Table 3. Back

Proposition 41. *Let Γ be a context, P a unification problem in $\Lambda_{dB}(\mathcal{X})$ well typed in Γ and, $\mathcal{A}(P)$ its unification tree. For each problem P_α in $\mathcal{A}(P)$, there exists a unification system P^*, derived from P_F using* **Unif**, *such that:*

1. *if P_α contains a branch that leads to a success node, then there exists a derivation of P^* that leads to a solved form;*
2. *if P_α contains a branch that leads to a fail node, then there exists a derivation of P^* that leads to a fail node;*

3. *if P_α is formed by the equations $eq_1, \ldots, eq_s$ that are well typed in context Γ, then there exists a unification system P_B^*, derived from P^* using the strategy* **Back**, *containing the equations $eq_{1_F}, \ldots, eq_{s_F}$ well typed in context Γ, up to the renaming of meta-variables. Moreover, any other equation in P_B^* is either flexible-flexible or solved.*

Proof. 1. Suppose P_α contains a branch with a success node and let $P_{\alpha\gamma}$ be the father of this success node. We consider two cases:

• *The success node is obtained after an application of SIMPL to $P_{\alpha\gamma}$*: In this case, $P_{\alpha\gamma}$ contains (at least) one rigid-rigid equation of the form:

$$\lambda_{A_1} \ldots \lambda_{A_n}.\underline{k}\ e_1^1 \ldots e_p^1 =^? \lambda_{A_1} \ldots \lambda_{A_n}.\underline{k}\ e_1^2 \ldots e_p^2 \tag{19}$$

well typed in context Γ and, where the terms $e_1^1, \ldots, e_p^1, e_1^2, \ldots, e_p^2$ are flexible or, if for some $1 \leq i \leq p$ and $1 \leq j \leq 2$, e_i^j is not flexible then $e_i^1 = e_i^2$; otherwise, the resulting problem could not be a success node. After the simplification step every rigid-rigid equation is replaced by a finite number of equations of the form flexible-flexible or trivial and, hence $\overline{P}_{\alpha\gamma}$ is a success node.

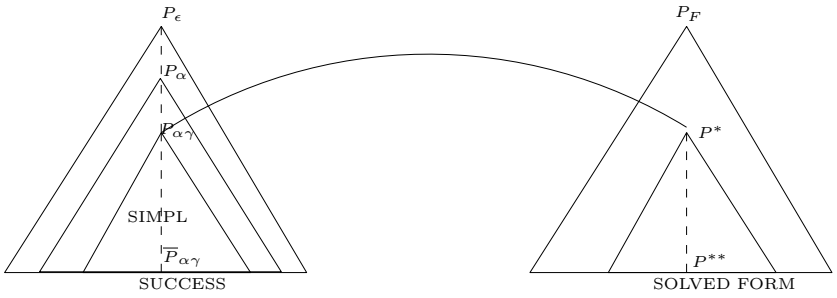


Fig. 13.

According to Lemma 40, there exists a unification system P^* derived from P_F for the problem $P_{\alpha\gamma}$ (see Fig. 13). In particular, for each equation of the form of equation (19), there exists an equation in P^* of the form

$$\underline{k}\ \overline{e}_1^1 \ldots \overline{e}_p^1 =^?_{\lambda\sigma} \underline{k}\ \overline{e}_1^2 \ldots \overline{e}_p^2 \tag{20}$$

well typed in context $A_1 \cdot \ldots \cdot A_n \cdot \Gamma$ and such that the terms $\overline{e}_1^1, \ldots, \overline{e}_p^1, \overline{e}_1^2, \ldots, \overline{e}_p^2$ are flexible or, if for some $1 \leq i \leq p$ and $1 \leq j \leq 2$, $\overline{e}_i^j$ is not flexible then $\overline{e}_i^1 = \overline{e}_i^2$. After an application of **Dec-App**, the equation (20) is replaced by a finite number of equations of the form flexible-flexible or trivial. This process is repeated for each rigid-rigid equation in P^* that, after all will have only flexible-flexible equations and, hence is in solved form.

• *The success node is obtained after an application of MATCH to* $P_{\alpha\gamma}$: In this case, $P_{\alpha\gamma}$ contains (at least) one flexible-rigid equation. Let $P_{\alpha\gamma k}$ $(k > 0)$ be the considered success node. Notice that if $P_{\alpha\gamma}$ contains more than one flexible-rigid equation then all these equations have the same meta-variable as the head of the flexible term; otherwise the resulting problem could not be a success node. Moreover, all possible flexible-rigid equations have the form:

$$\lambda_{A_1} \dots \lambda_{A_n}.X\ e_1 \dots e_p =^? \lambda_{A_1} \dots \lambda_{A_n}.\underline{h} \tag{21}$$

well typed in context Γ. If the application of MATCH corresponds to an imitation step then $h > n$ and, since the generated substitution does not introduce new meta-variables, the resulting equation is trivial.
From Lemma 40, there exists a unification system P^* derived from P_F and containing an equation of the form:

$$Y[\overline{e}_p \cdot \dots \cdot \overline{e}_1 \cdot \uparrow^n] =^?_{\lambda\sigma} \underline{h}$$

well typed in context $A_1 \cdot \dots \cdot A_n \cdot \Gamma$. Since $h > n$, after an application of **Exp-App** followed by an application of **Replace** and **Normalise** we get a solved form.
If the application of MATCH corresponds to a projection step then all equations of the form of equation (21) have the argument over which the projection is equal to $\underline{h}$. That way, after a normalization we get a success node.
From Lemma 40, there exists a unification system P^* derived from P_F, and containing an equation of the form

$$Y[\overline{e}_p \cdot \dots \cdot \overline{e}_1 \cdot \uparrow^n] =^?_{\lambda\sigma} \underline{h}$$

well typed in context $A_1 \cdot \dots \cdot A_n \cdot \Gamma$. After an application of **Exp-App** followed by an application of **Replace** and **Normalise** every flexible-rigid equation reduces to trivial equations and, we get a solved form.

2. Suppose that P_α contains a branch that leads to a fail node. The unification step just before getting this node must be an application of SIMPL to the father of this fail node. We call $P_{\alpha\gamma}$ the father node that must contain a rigid-rigid equation with different heads:

$$\lambda_{A_1} \dots \lambda_{A_n}.\underline{k}\ e^1_1 \dots e^1_{p_1} =^? \lambda_{A_1} \dots \lambda_{A_n}.\underline{h}\ e^2_1 \dots e^2_{p_2}$$

well typed in context Γ and such that $h \neq k$.
From Lemma 40, there exists a unification system P^* derived from P_F and containing the equation

$$\underline{k}\ \overline{e}^1_1 \dots \overline{e}^1_{p_1} =^?_{\lambda\sigma} \underline{h}\ \overline{e}^2_1 \dots \overline{e}^2_{p_2}$$

well typed in context $A_1 \cdot \dots \cdot A_n \cdot \Gamma$. After an application of **Dec-Fail** we get a fail node.

3. Suppose that $P_\alpha = eq_1 \wedge \ldots \wedge eq_s$ $(s > 0)$. It is enough to prove that, for an arbitrary equation eq_j $(1 \leq j \leq s)$ of P_α, we can obtain eq_{j_F} from the unification system P^* given by Lemma 40 via the strategy **Back**. In fact, the strategy **Back** does not propagate changes to other equations because it does not involves substitution. The proof is divided according to the structure of the equation eq_j:

   • *eq_j is a flexible-rigid equation*: In this case, eq_j has the form:

   $$\lambda_{A_1} \ldots \lambda_{A_n}.X\ e_1^1 \ldots e_{p_1}^1 =^? \lambda_{A_1} \ldots \lambda_{A_n}.\underline{h}\ e_1^2 \ldots e_{p_2}^2$$

   well typed in context Γ, where X is a meta-variable of type $B_1 \to \ldots \to B_{p_1} \to A$ (A atomic). According to Lemma 40, there exists a unification system P^* derived from P_F containing an equation of the form:

   $$Y[\overline{e}_{p_1}^1 \cdot \ldots \cdot \overline{e}_1^1 \cdot \uparrow^n] =^?_{\lambda\sigma} \underline{h}\ \overline{e}_1^2 \ldots \overline{e}_{p_2}^2$$

   well typed in context $A_1 \cdot \ldots \cdot A_n \cdot \Gamma$, where Y is a meta-variable of type A. The context of Y is given by $B_{p_1} \cdot \ldots \cdot B_1 \cdot \Gamma$. After an application of **Anti-Exp-λ** we get a system containing the conjunction:

   $$Y[\overline{e}_{p_1}^1 \cdot \ldots \cdot \overline{e}_1^1 \cdot \uparrow^n] =^?_{\lambda\sigma} \underline{h}\ \overline{e}_1^2 \ldots \overline{e}_{p_2}^2 \wedge Y =^?_{\lambda\sigma} Z[\uparrow]\ \underline{1}$$

   where Z is a fresh meta-variable of type $B_{p_1} \to A$ and context $B_{p_1-1} \cdot \ldots \cdot B_1 \cdot \Gamma$. After an application of **Replace** and then **Normalise** we get another unification system containing the conjunction:

   $$Z[\overline{e}_{p_1-1}^1 \cdot \ldots \cdot \overline{e}_1^1 \cdot \uparrow^n]\ \overline{e}_{p_1}^1 =^?_{\lambda\sigma} \underline{h}\ \overline{e}_1^2 \ldots \overline{e}_{p_2}^2 \wedge Y =^?_{\lambda\sigma} Z[\uparrow]\ \underline{1}.$$

   Repeating $p_1 - 1$ times the same strategy we get a unification system containing the equation:

   $$W[\uparrow^n]\ \overline{e}_1^1 \ldots \overline{e}_{p_1}^1 =^?_{\lambda\sigma} \underline{h}\ \overline{e}_1^2 \ldots \overline{e}_{p_2}^2$$

   well typed in context $A_1 \cdot \ldots \cdot A_n \cdot \Gamma$. After n applications of **Anti-Dec-λ** we get:

   $$\lambda_{A_1} \ldots \lambda_{A_n}.W[\uparrow^n]\ \overline{e}_1^1 \ldots \overline{e}_{p_1}^1 =^?_{\lambda\sigma} \lambda_{A_1} \ldots \lambda_{A_n}.\underline{h}\ \overline{e}_1^2 \ldots \overline{e}_{p_2}^2$$

   well typed in context Γ. Now we apply **Anti-Exp-λ** whenever it is possible to every sub-term $\overline{e}_j^i$ $(1 \leq i \leq 2; 1 \leq j \leq p_1)$ of this equation and we get $e_{j_F}^i$ up to the renaming of meta-variables. In this way, the current unification system contains an equation equal to eq_{j_F} up to the renaming of meta-variables.
   • *eq_j is a rigid-rigid equation*: In this case, eq_j has the form

   $$\lambda_{A_1} \ldots \lambda_{A_n}.\underline{k}\ e_1^1 \ldots e_{p_1}^1 =^? \lambda_{A_1} \ldots \lambda_{A_n}.\underline{h}\ e_1^2 \ldots e_{p_2}^2$$

   well typed in context Γ. According to Lemma 40, there exists a unification system P^* derived from P_F and containing an equation of the form

   $$\underline{k}\ \overline{e}_1^1 \ldots \overline{e}_{p_1}^1 =^? \underline{h}\ \overline{e}_1^2 \ldots \overline{e}_{p_2}^2$$

well typed in context $A_1 \cdot \ldots \cdot A_n \cdot \Gamma$. Applying **Anti-Exp-$\lambda$**, **Replace** and **Normalise** whenever it is possible to every sub-term $\overline{e}^1_1, \ldots, \overline{e}^1_{p_1}, \overline{e}^2_1, \ldots, \overline{e}^2_{p_2}$ and applying n times the rule **Anti-Dec-λ** to the above equation, we get a unification system that contains an equation equal to eq_{j_F} up to the renaming of meta-variables.

• *eq_j is a flexible-flexible equation*: In this case, eq_j has the form

$$\lambda_{A_1} \ldots \lambda_{A_n}.X\ e^1_1 \ldots e^1_{p_1} =^? \lambda_{A_1} \ldots \lambda_{A_n}.Y\ e^2_1 \ldots e^2_{p_2}$$

well typed in context Γ, where X is a meta-variable of type $B_1 \to \ldots \to B_{p_1} \to A$ (A atomic). According to Lemma 40, there exists a unification system P^* derived from P_F and containing the equation

$$Z[\overline{e}^1_{p_1} \cdot \ldots \cdot \overline{e}^1_1 \cdot \uparrow^n] =^?_{\lambda\sigma} W[\overline{e}^2_{p_2} \cdot \ldots \cdot \overline{e}^2_1 \cdot \uparrow^n]$$

well typed in context $A_1 \cdot \ldots A_n \cdot \Gamma$, where Z and W are meta-variables of type A. Repeating the strategy used in the previous item to all sub-terms of this equation and then applying n times the rule **Anti-Dec-λ**, we get a unification system containing an equation equal to eq_{j_F} up to the renaming of meta-variables.

During the unification process in the $\lambda\sigma$-calculus, new equations are introduced after applications of **Exp-λ**, **Exp-App** or **Anti-Exp-λ**. These equations are of the form $X =^?_{\lambda\sigma} a$, where X is a meta-variable and a is a term without occurrences of X. After an application of **Replace** every occurrence of X in the unification system will be replaced by a and this equation becomes solved. It will remain solved during the whole process because no rule applies to X, although the term a can change. □

The next example illustrates the contents of Proposition 41.

Example 42. Consider the unification problem P presented in Example 26 whose unification tree $\mathcal{A}(P)$ is given in Fig. 4. Consider, for instance, the subgoal

$$\overline{P}_1 = \{\lambda_{B\to B}.X_1\ \underline{3} =^?\ \lambda_{B\to B}.\underline{4}\ \underline{3}\}$$

which is well typed in context $\Gamma = \{A \to B \cdot A \cdot A \to A \cdot nil\}$ and whose unification tree contains one fail node and two success nodes. The proof of Lemma 40 is constructive and lead us to the unification system

$$P^* = \{H_1[\underline{3} \cdot \uparrow] =^?_{\lambda\sigma} \underline{4}\ \underline{3} \wedge X =^?_{\lambda\sigma} \lambda_A.\underline{2}\ H_1 \wedge Y =^?_{\lambda\sigma} \underline{2}\ H_1\}$$

whose derivation tree also contains one fail node and two success nodes (see Fig. 5). Applying the strategy **Back** to P^* we get the unification system:

$$P^*_B = \{\lambda_{B\to B}.N[\uparrow]\ \underline{3} =^?_{\lambda\sigma} \lambda_{B\to B}.\underline{4}\ \underline{3}\ \wedge\ X =^?_{\lambda\sigma} \lambda_A.\underline{2}\ (N[\uparrow]\ \underline{1}) \wedge Y =^?_{\lambda\sigma} \underline{2}\ (N[\uparrow]\ \underline{1}) \wedge H_1 =^?_{\lambda\sigma} N[\uparrow]\ \underline{1}\}$$

where the equation

$$\lambda_{B\to B}.N[\uparrow]\ \underline{3} =^{?}_{\lambda\sigma} \lambda_{B\to B}.\underline{4}\ \underline{3}$$

corresponds to the precooking translation of the equation

$$\lambda_{B\to B}.X_1\ \underline{3} =^{?} \lambda_{B\to B}.\underline{4}\ \underline{3}$$

up to the renaming of meta-variables and all the other equations are solved. The solution σ of $\overline{P}_1$ is given by $\sigma = \{X_1/\lambda_A.\underline{4}\ \underline{1}, X_1/\lambda_A.\underline{4}\ \underline{3}\}$. It is easy to check that the grafting $\sigma_F = \{N \mapsto \lambda_A.\underline{4}\ \underline{1}, N \mapsto \lambda_A.\underline{4}\ \underline{3}\}$ obtained from σ, after renaming X_1 to N, is a solution to P^*_B.

Corollary 43. *Let P be a unification problem in the simply typed λ-calculus and $\mathcal{A}(P)$ its unification tree. For each unification problem P_α in $\mathcal{A}(P)$ with solution σ there exists a unification system P^*_B derived from P_F using the strategies* **Unif** *and* **Back** *that has σ_F as solution after an adequate renaming of meta-variables.*

Proof. Let Γ be a context and P_α be a unification problem in $\mathcal{A}(P)$ well typed in context Γ. From Proposition 41, we know that there exists a derivation from P_F using **Unif** and **Back** that generates a unification system P^*_B that contains all the equations in P_{α_F} up to renaming of meta-variables. Moreover, all the other equations in P^*_B are solved and, hence the grafting σ_F, after an adequate renaming of meta-variables, is a solution to P^*_B according to Proposition 3.5 of [DHK00].

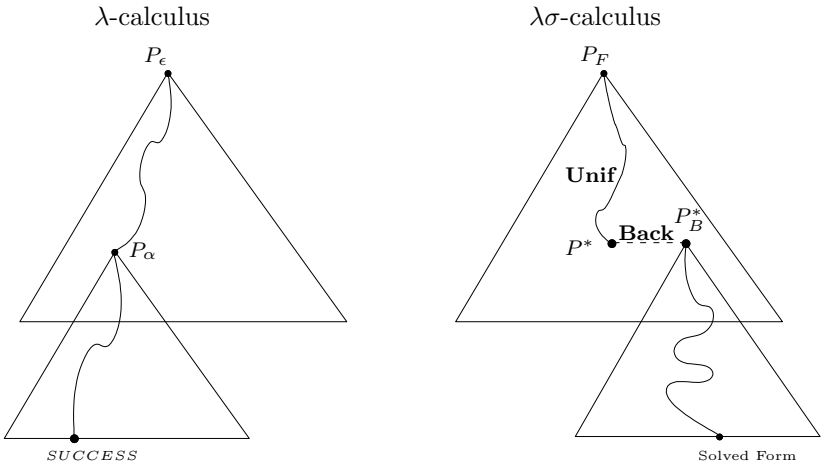


Fig. 14. General Unification Scheme in the λ- and $\lambda\sigma$-calculus

Figure 14 shows the general scheme that relates unification in the λ-calculus and in the $\lambda\sigma$-calculus. □

Corollary 43 allow us to conclude that unification in the simply typed $\lambda\sigma$-calculus is a generalization of Huet's algorithm since every solution for a unification system computed in the $\lambda\sigma$-calculus is also computed by Huet's algorithm.

5 Conclusions and Future Work

In a stepwise fashion, we compared two different styles of HOU: the classical HOU for the simply typed λ-calculus of Huet [Hue75] and HOU via the simply typed $\lambda\sigma$-calculus [DHK00]. The contributions of this paper are:

- We enriched the *matching trees* of Huet by introducing a new structural notation called *unification tree*. This notation was essential to provide a precise presentation of the derivations of Huet's algorithm and, constituted an important tool for establishing the structural correspondence between HOU à la Huet and HOU via explicit substitutions.
- Although it is a straightforward translation of Huet's HOU algorithm, we explicitly introduced Huet's HOU algorithm for the simply typed λ-calculus in de Bruijn's notation. This was done in order to simplify the comparison between Huet's HOU algorithm and the $\lambda\sigma$-HOU method, since the latter uses the de Bruijn's notation.
- Both the simply typed λ-calculus with names and in de Bruijn's notation include meta-variables. Although the use of meta-variables is not essential for the unification methods treated here, its use simplifies their presentation and allows us to keep a clear difference between substitutions generated by applications of β-reductions and substitutions generated by the unification rules. The difference between typing meta-variables in the λ-calculus and in the $\lambda\sigma$-calculus was emphasized through examples since the unification mechanism in the former uses (higher-order) substitution while the latter uses grafting (first-order substitution).
- Unification derivations in the simply typed $\lambda\sigma$-calculus were presented in a tree structure notation called *derivation tree* which jointly with the unification tree structure permits a better visualization of the relations between unification derivations in both methods.
- By using these structures, we proved that Huet's HOU and the $\lambda\sigma$-HOU preserve an important structural relation between (sub-)problems: For a given unification problem P in the simply typed λ-calculus, we have that for each (sub-)problem of P in the unification tree $\mathcal{A}(P)$, there exists a counterpart in the derivation tree of P_F. This allow us to conclude that the $\lambda\sigma$-HOU is a generalization of Huet's algorithm and that solutions computed by the latter are always computed by the former method.

We believe that this structural comparison is important to provide a better understanding of HOU methods based on explicit substitutions and to shed some light on questions related to practical and implementational issues as well as on the whole of explicit substitutions in higher-order unification.

Natural extensions of this work include considering an optimized implementation of the $\lambda\sigma$-HOU algorithm based on the ideas behind the notion of the pseudo-precooking that in fact combines the precooking with some unification rules. In addition, these ideas can be extended to other styles of explicit substitutions like the λs_e-calculus and the suspension calculus.

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